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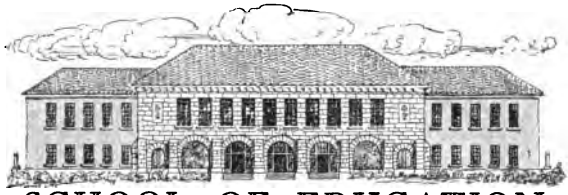

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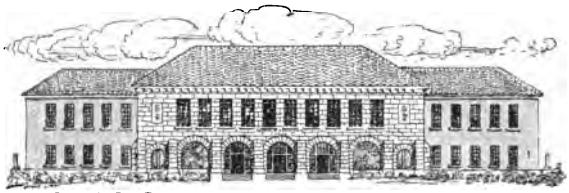

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PLANE GEOMETRY

BY

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PREFACE

As a result of the widespread discussion during recent years on the improvement of our courses in elementary geometry, the majority of thoughtful teachers appear to have reached substantial agreement on at least one point: To begin the course in plane geometry in the traditional formal manner is pedagogically irrational and scientifically unnecessary. There has accordingly arisen an increasing demand for a textbook which will supply a pedagogically rational approach to the study of Plane Geometry, without sacrificing the logical structure of the subject.

The present text aims to supply this demand. The beginning should be thoroughly concrete, informal, and to the pupil natural and interesting. More formal methods should be introduced gradually at a time when the pupil can understand their significance and value. These ends we have sought to attain by making a systematic study of geometric drawing the basis of the first chapter. Throughout this chapter points, lines, etc., are concrete things to be drawn with a pencil or a piece of chalk; the reasoning involved is couched in easy, natural language, without any of the stiffness of a formal arrangement. While the development of the subject matter of this chapter is wholly systematic, the emphasis is intended to be placed on the exercises which have been carefully selected both with respect to interest and for drill in developing the power of making inferences. The study of this first chapter will, we confidently believe, accomplish two ends: (1) It will lead the pupil to gain a thorough understanding of the fundamental notions of geometry; (2) it will develop in him the power of attack and insight into the spirit of a geometric problem. Beginning with more formal methods in the second chapter,

he will be in a position to pursue the remainder of the course with understanding, pleasure, and profit. Our confidence in this respect is not based merely on theoretical considerations, but rather on the fact that this method of approach has been used for several years with most satisfactory results in actual class work with large numbers of pupils.

Among other more or less characteristic features of our text we would mention briefly the following:

(1) *The systematic use of symmetry as a method of proof.* That this can be made a useful and active means of inference may be seen, for example, by reference to pages 10 to 17. Frequently where a rather long process of reasoning by congruence yields but a single detail of a figure, the method of symmetry enables one to grasp the whole relationship by one mental act. Above all other advantages, however, this method of inference has shown itself by actual classroom experience to appeal strongly to the pupil as in his eyes a natural method of procedure.

(2) *The disposal of the so-called incommensurable cases on purely practical grounds,* but in a way which is entirely in the spirit of the rigorous logical development.

(3) *The introduction of the definitions and elementary applications of the trigonometric ratios,* without lengthening the course. This has been made possible by the omission (or insertion in the exercises) of a number of theorems hitherto retained which are not necessary links in the logical development or do not serve a useful purpose as basal theorems. The introduction of this material seems to us highly desirable from whatever point of view the study of geometry in secondary schools be advocated, whether cultural, disciplinary, or utilitarian.

(4) *The two-color printing of most of the figures.* The device of printing the auxiliary lines and construction lines in a subdued color we hope will prove a welcome innovation, tending to emphasize the distinction and improving the appearance of the page.

We have taken great care in the selection and grading of the numerous exercises, and have made a consistent effort throughout to develop independence and resourcefulness on the part of the

pupils. Analysis is continually resorted to or suggested in the solution of problems, and genetic methods are given the preference over didactic in the proof of theorems wherever possible.

In conclusion, it is a pleasure to acknowledge our indebtedness to such excellent works as Borel's "Géométrie," the "Traité de Géométrie," by Rouche et Comberousse, "Ebene Geometrie" by Mahler, "Congruent Figures" by Henrici, and many other texts printed in this country and abroad. Our cordial thanks are due also to several friends who have taken a live interest in our work, and especially to Dr. F. M. Morgan of Dartmouth College, who, in reading portions of the manuscript and in rendering valuable assistance in seeing the book through the press, has found occasion to make a number of helpful suggestions and criticisms.

J. W. Y.,
A. J. S.

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SYMBOLS AND ABBREVIATIONS USED IN GEOMETRY

$=$, is equal to.		*R.*,	right angle.
$\equiv$, is congruent to.		*Def.*,	definition.
$>$, is greater than.		*F.P.*,	fundamental proposition.
$<$, is less than.		*Ax.*,	axiom.
$\therefore$, therefore.		*Hyp.*,	hypothesis.
$\because$, since.		*Cor.*,	corollary.
$\perp$, perpendicular, or is perpendicular to.		*Ex.*,	exercise.
		Const.,	construction.
$\perp$s, perpendiculars.		*Iden.*,	identity.
$\angle$, angle, $\angle$s, angles.		*Subst.*,	substitution.
$\parallel$, parallel, or is parallel to.		*Adj.*,	adjacent.
$\parallel$s, parallels.		*Sup.*,	supplementary.
$\odot$, circle, $\odot$s, circles.		*Ext.*,	exterior.
$\triangle$, triangle, $\triangle$s, triangles.		*Int.*,	interior.
Rt. $\triangle$, right triangle.		*Alt.*,	alternate.
$\square$, parallelogram.		*Pt.*,	point.
$\square$s, parallelograms.		*Pts.*,	points.
rt., right, *st.*, straight.		*e.g.*,	for example.

The signs $+$, $-$, $\times$, $\div$ have the same meaning as in algebra and arithmetic.

GREEK LETTERS FREQUENTLY USED IN NAMING ANGLES

α, pronounced alpha.	δ, pronounced delta.
β, pronounced beta.	θ, pronounced theta.
γ, pronounced gamma.	ϕ, pronounced phi.

PLANE GEOMETRY

CHAPTER I

FUNDAMENTAL NOTIONS AND ELEMENTARY CONSTRUCTIONS

THE STRAIGHT LINE

Use of the Straightedge

1. If we wish to draw a straight line, we require (1) a *plane* (*flat surface*) on which to draw the line, and (2) a *straightedge* for a guide. The plane may be represented by the surface of a blackboard or by the surface of a sheet of paper lying on a table; and the straightedge by the edge of an ordinary ruler, T-square, or draftsman's triangle (Figs. 1 and 2).

Fig. 1.

2. Test of a straightedge. Through two convenient points A and B, trace a line on a sheet of paper or other flat surface, using the edge of the ruler, or other instrument to be tested as a guide. Then reverse the ruler end for end (keeping the same side uppermost) and again trace a line through the points A and B. If the two traces are identical whatever be the position of the ruler, provided only the traces pass through the given points, the edge of the ruler is straight. This test depends on the following fundamental properties of a straight line and plane.

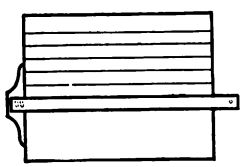


Fig. 2.

3. FUNDAMENTAL PROPOSITION. *Through any two distinct points it is possible to draw one straight line, and only one.*

We may, therefore, say that two distinct points *determine* a straight line.

4. Notation. Points are usually designated by the capital letters of the alphabet. Thus we speak of the "point A," the "point P," etc.

A straight line may then be conveniently designated, or labeled, by the letter representing any two points of the line. Thus the "line AB" means the line passing through the two points A and B.

5. *If a line CD has two points in common with a line AB, it coincides with AB.* For by § 3 AB and CD are the same line.

6. *Two distinct straight lines cannot have more than one point in common.* For if they have more than one point in common, they are the same line, § 3.

7. FUNDAMENTAL PROPOSITION. *If a straight line has two distinct points in common with a plane, it lies wholly in the plane.*

8. A line segment. The part of a straight line lying between two of its points is called a *line segment.* A line segment is designated by its end points, or by a small letter, in the figure usually placed near the middle of the segment. We thus speak of the "segment AB" or the "segment a."

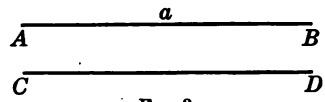


FIG. 3.

9. Equal segments. The line segments AB and CD are said to be *equal* if they can be so placed that their end points coincide.

10. PROBLEM. *On a given straight line to construct a line segment equal to a given line segment.*

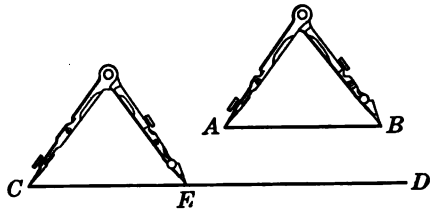


FIG. 4.

Construction. Adjust the dividers so that the legs rest on the end points of the given segment. Then set the legs of the dividers on the given straight line. The segment of the latter included by the legs of the dividers is equal to the given segment.

11. Unequal segments. Given two segments AB and CD. Imagine the segment AB placed along the segment CD so that the end point A coincides with the end point C. If then the point B lies between C and D, the segment AB is said to be *less* than the segment CD. If the point B lies beyond the point D, the segment AB is said to be *greater* than the segment CD. The former relation is written $AB < CD$; the latter, $AB > CD$.

12. Sum of two segments. If A, B, C are three points on a straight line and B is *between* A and C, the segment AC is the *sum* of the segments AB and BC; in symbols $AB + BC = AC$.

13. A broken line. A line which is not straight but which is composed of line segments placed end to end is called a *broken line.*

14. A curved line. A line no part of which is straight is called a *curved line.*

15. FUNDAMENTAL PROPOSITION. *The shortest path between two points is the line segment joining the points.*

16. A geometric figure. Any combination of points and lines in a plane is called a *plane geometric figure*, or simply a plane figure. It is the object of plane geometry to study the properties of plane figures. All points and lines considered in this book are supposed to be in the same plane, unless specifically stated otherwise.

17. EXERCISES

1. Take four points A, B, C, D, and draw the straight lines determined by any pair of these. Show that there are, in general, six of these lines. Under what conditions are there less than six?

2. Take four straight lines AB, CD, EF, GH, and mark the intersection of any pair of these. Show that there are, in general, six of these points. Under what conditions are there less than six?

3. Take a straight line AB, and two segments a and $b (a > b)$. Upon AB lay off a segment equal to $a + b$; also lay off one equal to $a - b$.

4. Take a broken line $MNOPQR$. Upon a given straight line lay off a segment equal in length to the broken line.

5. A spider in an upper corner of a room of given dimensions observes a fly in the lower diagonally opposite corner, and immediately proceeds to attack his prey in the most direct route possible for the spider. Determine the path of the spider. Represent the room by a card board box and mark the path on the walls and floor or ceiling.

THE CIRCLE

Use of the Compasses

18. The circle. Imagine a line segment OA to rotate in a plane about the point O until it returns to its original position. The path of the point A is then a *circle*. The point O is called the *center* of the circle.

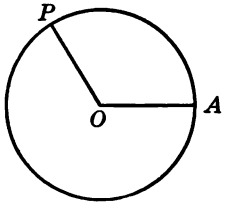


Fig. 5.

19. From this definition we observe immediately that *all the points of a circle are at the same distance from its center.*

20. The radius. The distance of any point on a circle from its center is called *the radius* of the circle. Also any *line segment* as OP joining the center to any point on the circle is called *a radius.*

NOTE. It will be observed that the word radius is used to denote either a *line segment* OP, or the *length* of any such segment. This use of the word in two senses does not, however, lead to confusion, since the context always makes clear which sense is meant.

21. To describe a circle. *To draw* or *describe* a circle use is made of *compasses.* A circle with center O and radius OA is said to be *described from O as a center with the radius OA.*

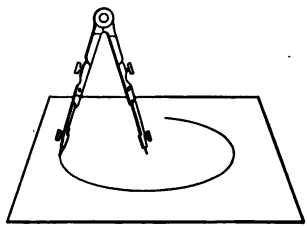


FIG. 6.

22. FUNDAMENTAL PROPOSITION. *In a plane, from a given center and with a given radius one circle, and only one, can be described.*

We may express this also by saying that a circle in a plane is *determined* when its center and its radius are given.

23. Arcs and chords. Two points on a circle divide the circle into two parts called *arcs.* A line segment joining two points on a circle is called a *chord.* If a chord passes through the center of the circle it is called a *diameter.* The extremities of any diameter divide the circle into two parts each of which is called a *semicircle.*

24. *All radii, and all diameters, of a circle are respectively equal.*

25. All points in the plane of a circle whose distance from the center is less than the radius are said to be *inside* the circle; and all points of the plane whose distance from the center is greater than the radius are said to be *outside* the circle. (All points in the plane whose distance from the center is equal to the radius are, of course, *on* the circle.)

26. FUNDAMENTAL PROPOSITIONS. Let the student draw a circle and then draw a straight line through any point inside the circle; also let him draw two intersecting circles. The following properties of circles which are useful in investigating the relative positions of circles and straight lines will then be readily admitted.

I. *A straight line in the plane of a circle and passing through any point inside the circle cuts the circle in two, and only two, distinct points.*

II. *If two circles intersect in a point which is not on the line joining their centers, they intersect in a second point, and no more.*

27. **EXERCISES**

1. At the entrance to New York harbor is a gun having a range of 12 miles. Draw a plan representing the surface exposed to the fire of the gun, using a scale $\frac{1}{16}$ inch to a mile.

2. Two forts are placed on opposite sides of a harbor entrance 7 miles apart. Each has a gun having a range of 5 miles. Draw a figure representing the surface in which an enemy is exposed to the fire of both guns, using a scale $\frac{1}{4}$ inch to a mile.

3. A donkey is tethered to a stake near a long straight hedge by a rope 36 feet long. Draw a plan showing his complete range, if he can nibble just 48 feet of hedge. Use a scale 1 inch to 24 feet.

4. Take a straight line AB, a line segment s, and a point P. Find a point X on AB at a distance s from P. Under what conditions is the problem impossible? How many solutions can it have?

5. Take two points P and Q, and two line segments r and s. Find a point X at a distance r from P and a distance s from Q. Is this problem ever impossible? How many points X may be found?

6. Take two points P and Q, and a line segment r. Find a point X at a distance r from P and Q. Discussion.

7. With a given radius r describe a circle passing through two given points P and Q. Discussion.

8. The following figures represent the geometric bases of three types of arches used in architecture called, respectively, the *equilateral arch*, the *segmental arch*, and the *Moorish arch*.

Construct Fig. (a), given the span $AB = 1\frac{1}{2}$ inches.

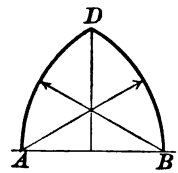


FIG. (a).

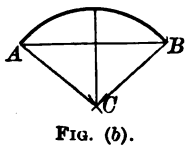


FIG. (b).

Construct Fig. (b), given the span $AB = 1\frac{1}{2}$ inches, and the radius $CA = 1$ inch.

Construct Fig. (c), given the span $AB = 1\frac{1}{2}$ inches, and the radius $CA = \frac{7}{8}$ inch.

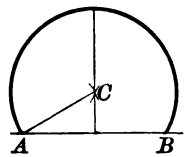


FIG. (c).

MOTION

Congruent Figures

28. Motion. When we defined a circle we imagined a line segment to *move* in a certain way (§ 18). Also in discussing the equality of line segments the idea of motion was implied (§ 9, see also § 11). The fact that a figure may be thought of as moving from one position to another is of such fundamental importance in the study of geometry, that it is desirable that we gain a clear idea of the motions that are possible. We will readily agree to the following:

29. Fundamental Propositions. I. *Any figure in a plane may be imagined to slide along the plane in such a way that every point in the figure moves along a straight line.*

30. II. *Any point C of the plane of a figure may be kept fixed and the figure imagined to rotate in the plane about C as a center. Every point of the figure (except C) will then describe a circle with center C.*

31. III. *Any two points S and T of the plane of a figure may be kept fixed and the plane imagined to rotate about the line ST as an axis, until it coincides again with its original position, but with its sides reversed.*

We shall describe this type of motion by saying that the figure has been *turned over* about the *axis ST. During such a motion every point of the axis remains fixed.*

32. Congruence of figures. On the idea of motion depends a fundamental relation which may exist between two figures and which is defined as follows: Two figures are said to be *identically*

equal or *congruent*, if by a suitable motion the figures can be placed so as to coincide throughout.

33. If two figures in the same plane are congruent, they may be made to coincide by the successive application of one or more of the types of motion I, II, III, described in §§ 29, 30, 31. The following propositions will illustrate the truth of this statement.

34. *Two circles with equal radii are congruent.* Let the circles be in the same plane with centers C and C'. Imagine the first circle to slide so that its center C moves along the straight line CC' until it coincides with C'. Then, since their radii are equal, the circles will coincide throughout (§ 22). Here a slide was sufficient to show the figures congruent.

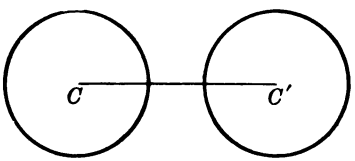


FIG. 7.

35. If we wish to bring two equal line segments, AB and CD, into coincidence, we need only imagine AB to slide so that A moves along the line AC until A coincides with C, and then

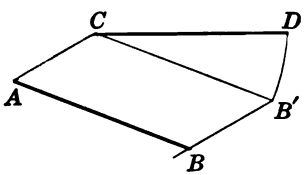


FIG. 8.

imagine AB to rotate about C until B coincides with D. Here a slide followed by a rotation suffices.

Cases where the third type of motion is required to establish congruence will appear almost immediately.

SYMMETRY WITH RESPECT TO A STRAIGHT LINE

36. A symmetric figure. Imagine Fig. 9 to rotate about the straight line MN as an axis until it comes again into the plane of its original position. Let Fig. 10 be a duplicate of Fig. 9 together with the latter after the rotation. Figure 10 is said to be *symmetric* with respect to the axis MN. It should be observed that Fig. 10 has the property that, if it is turned over on the axis MN, it will coincide throughout with its original position.

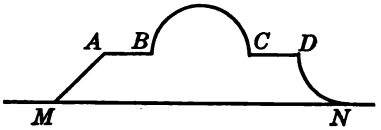


FIG. 9.

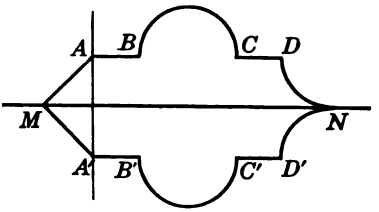


FIG. 10.

37. Symmetry with respect to a straight line. In general, a figure is said to be *symmetric with respect to a straight line as an axis of symmetry*, if, after the figure has been turned over on the axis, it coincides throughout with its original position.

38. Corresponding parts of a symmetric figure. Any two parts of a symmetric figure are said to be *corresponding* which simply change places when the figure is turned over on the axis of symmetry. Either one of any two corresponding parts is called the *reflexion* of the other with respect to the axis of symmetry.

39. *If a figure is symmetric with respect to a straight line, corresponding parts are congruent* (§ 32).

40. *If a figure is symmetric with respect to a straight line, the lines joining corresponding points are symmetric with respect to the axis.*

For, if A, A' (Fig. 10) are two corresponding points, the latter will, when the figure is turned over on the axis, simply change places. The line AA' will therefore coincide with its original position after it has been turned over on the axis (§ 3).

41. Mid-point of a line segment. A point M of a line segment AB is called the *mid-point* of the segment, if the segments AM and MB are equal. The point M is then said to *bisect* the segment. Also any straight line through M is said to *bisect* the segment.

The following proposition will be readily admitted.

42. FUNDAMENTAL PROPOSITION. *The line segment joining two points on opposite sides of a straight line has a point in common with the line.*

43. *If a figure is symmetric with respect to a straight line, the line segments joining corresponding points are bisected by the axis of symmetry.* Thus in Fig. 10 the segment AA' is bisected by the axis MN (§ 39).

44. *A circle is symmetric with respect to any straight line passing through its center.*

For, if the circle be turned over on any straight line through the center, the latter remains fixed. The circle, after being turned over, then coincides with its original position, since it has the same center and the same radius (§ 22).

45 As a very useful consequence, we have the following:

The figure formed by any two circles is symmetric with respect to the line joining the centers of the two circles.

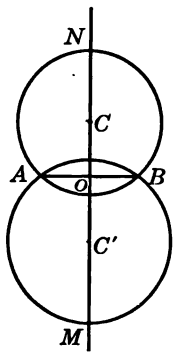


FIG. 11.

46. In connection with this proposition it should be noted that if two circles intersect in two points A, B, these two points are corresponding points in the symmetry with respect to the line joining the centers. The line AB is then symmetric with respect to this axis, and the segment AB will be bisected by the axis. This fact will be made the basis of the following important construction.

47. PROBLEM. *To bisect a given line segment AB.*

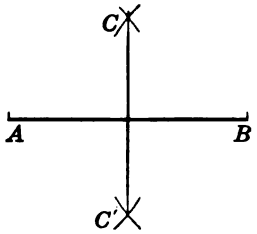


FIG. 12.

Construction. Reference to § 46 shows that the problem will be solved if we can draw two circles through A and B. To do this choose any convenient radius sufficiently long and from A and B as centers describe arcs of circles intersecting at C and C'. Join C to C'. Then the straight line CC' bisects AB.

For segments CA, CB, $C'A$, $C'B$ are equal, by construction. Therefore, two circles can be described from centers C and C' which will pass through A and B (§ 22).

48. As an immediate consequence of the preceding proof we infer the following: *If a straight line CC' is symmetric with respect to a straight line AB as an axis, then AB is symmetric with respect to CC' as an axis.*

49. EXERCISES

1. Take a circle with center C, a point P on the circle, and any straight line MN drawn through C. Find the reflexion of P with respect to the axis MN.

SUGGESTION: Reference to § 46 shows that the problem will be solved if we can obtain two circles passing through P and having their centers on MN. One such circle is given.

2. Take a straight line MN and a point P not on MN. Find the reflexion of P with respect to MN as an axis of symmetry.

3. If a figure is symmetric with respect to a straight line as an axis, show that lines joining any point on the axis to two corresponding points are corresponding lines.

SUGGESTION: Show that if the figure is turned over on the axis these lines will simply change places.

4. Take two lines AB and MN which meet at the point P. Find a line symmetrical to AB with respect to MN as an axis.

SUGGESTION: The reflexion of AB passes through P. Choose a point Q on AB and find its reflexion (Ex. 2).

5. If A, A' and B, B' are pairs of corresponding points in a symmetry with respect to a straight line as an axis, show that the lines AB and $A'B'$, and also AB' and $A'B$, are corresponding lines.

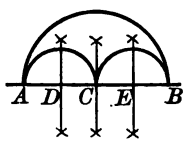
SUGGESTION: Imagine the figure to be turned over on the axis.

6. Take two straight lines AB and MN which do not meet within the limits of the drawing. Find a line symmetric to AB with respect to MN as an axis.

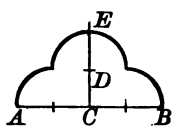
7. Take two straight lines AB and CD which meet at a point P. Find a line with respect to which AB and CD are symmetric.

SUGGESTION: The point P is on the axis. Describe a circle with center P. This will locate corresponding points on AB and CD. Then use § 43.

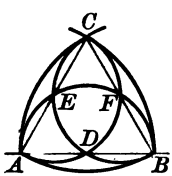
8. In the adjacent figure is shown the geometric basis of a design frequently used for tile flooring. $AC = CB$. Semicircles are constructed on AC, CB and AB as diameters. Construct the figure, using any segment AB as base.

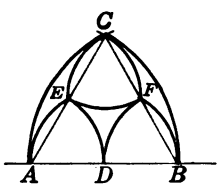


9. In the adjacent figure is shown the geometric basis of the round-headed trefoil arch. $AC = CB = CE$. CE is an axis of symmetry and D is the mid-point of CE. Construct the figure as shown, using any segment AB for the span.



10. In the following figures are shown the leading lines for several designs in window tracery. Construct the figures, using any segment AB as a base.





PERPENDICULAR LINES

50. Perpendicular lines. If a straight line MN is symmetric with respect to a straight line PQ, MN and PQ are called *perpendicular lines*, and each is said to be *perpendicular* to the other.

51. The result of § 40, may then be stated as follows: *The lines joining corresponding points of a figure symmetric with respect to a straight line are perpendicular to that line.*

52. In particular, *if two circles in the same plane intersect in two distinct points, the line through these points is perpendicular to the line joining the centers of the circles* (§ 46).

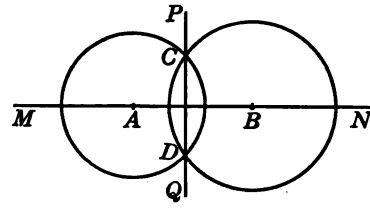


FIG. 13.

53. *Through any given point C not on a straight line MN, one straight line can be drawn perpendicular to MN, and only one.*

For, let D be the point on which C falls when the plane is turned over on MN. Any straight line through C perpendicular to MN must contain D. But the two points C and D completely determine a straight line (§ 3).

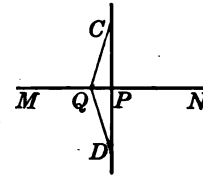


FIG. 14.

54. *The perpendicular is the shortest segment that can be drawn from a point to a straight line.*

For, in the preceding figure, let CQ be any other segment drawn from C to MN. Join Q to D. Then $CD < CQ + QD$ (§ 15). But $CP = DP$, and $CQ = QD$. (Why?) Therefore CP, the half of CD, is less than CQ, the half of $CQ + QD$.

55. Distance of a point from a straight line. The length of the perpendicular drawn from a point to a straight line is called the *distance of the point from the straight line.*

56. PROBLEM. *Through a given point C not on a given straight line MN, to draw the perpendicular to MN.*

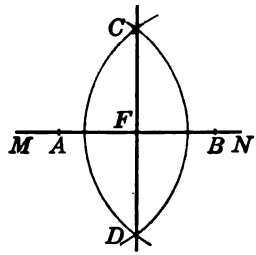


FIG. 15.

Construction. I. Reference to § 52 shows that the problem will be solved if we can draw two circles through C having their centers on MN. To do this choose any two convenient points A, B on MN as centers, and from these describe two circles through C (Fig. 15). Let these circles meet again in D. The line CD is the required line.

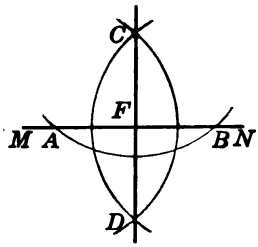


FIG. 16.

II. The preceding construction makes it necessary to set the compasses twice. This may be avoided as follows: With C as a center and a radius sufficiently long, describe a circle cutting the line MN in two points A, B (Fig. 16). Then the circles described from A and B as centers with the same radius will pass through C. The construction then proceeds as before.

57. PROBLEM. *Through a given point F on a straight line MN to draw a line perpendicular to MN.*

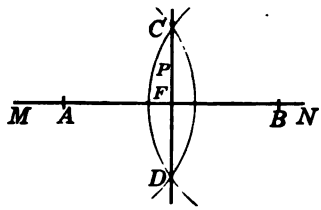


FIG. 17.

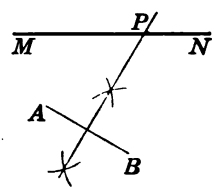
Construction. Take A and B two points on MN at the same distance from F (Fig. 17). From A and B as centers and a radius sufficiently long, describe circles intersecting in C and D. The line CD is the required line. Why?

58. From the symmetry of the preceding figure we easily infer the following important proposition: *Every point on the perpendicular drawn to a line segment at its mid-point is equally distant from the extremities of the segment.*

For, when the figure is turned over on CD, A and B exchange positions. Why? Let the student complete the proof.

59. **EXERCISES**

1. Take a straight line MN and two points A and B not on MN. Find a point on MN equally distant from A and B.



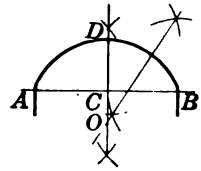
SOLUTION: Draw a perpendicular to the segment AB at its middle point. Let this line cut MN in P. P is the required point. (Why?)

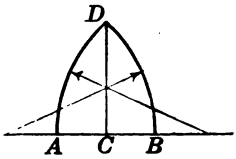
Is a solution ever impossible?

2. Take a circle with center C and two points A and B. Find a point on the circle equally distant from A and B.

Would a solution ever be impossible? How many solutions may there be?

3. Given AB, the span of a segmental arch, and CD its height. To draw the arch ADB.

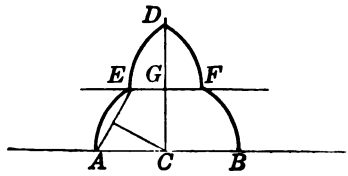


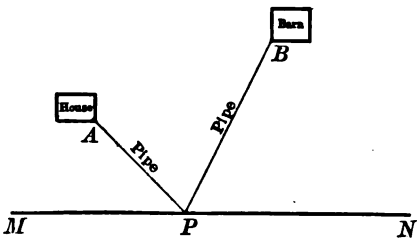


4. In the lancet arch the span is less than the height. To draw the arch, given the span AB and the height CD.

5. The obtuse arch is similar to the lancet arch except that the span is greater than the height. To draw the arch given the span AB and the height CD.

6. In the adjacent figure is shown a form of the pointed trefoil arch. The span AB is greater than the height CD. $EF = \frac{1}{2} AB$. To construct an arch similar to the figure shown, given the span AB and the height CD.





7. In the adjacent figure A represents the position of a house, B that of a barn and MN the edge of a river. Where must a pumping station P be located on the river bank so that the water can be piped to both house and barn with the least length of pipe?

HINT: Find the reflexion of B with respect to MN.

ANGLES

Use of the Draftsman's Triangle

60. A half line. The part of a straight line on one side of a point on the line is called a *half line.* Thus a point O on a straight line AB and between A and B divides it into two half lines OA and OB. The half lines OA and OB are said to *issue* from the point O, which is called the *origin.*

61. An angle. The figure formed by two half lines BA and BC which issue from the same point B is called an *angle.* The point B is called the *vertex* and the half lines BA and BC are called the *sides* of the angle.

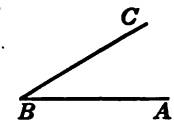


Fig. 18.

62. Notation. When an angle is isolated as in Fig. 18, it may be designated by merely naming the letter denoting its vertex. If, however, two or more angles have the same vertex as in Fig. 19, this method leads to ambiguity and a different method must be employed. In this case the angle may be designated by a small letter, usually a Greek letter, placed between the sides near the vertex. The angle may also be designated by the notation, angle ABC ($\angle ABC$), or angle CBA ($\angle CBA$), in which B denotes the vertex and A and C denote any other points taken on the sides, respectively. When naming an angle by three letters, it is important to observe that the letter denoting the vertex is always written between the other two.

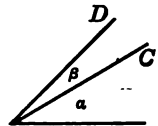


Fig. 19.

63. Equal angles. Two angles are said to be *equal* (*congruent*) if either can be placed on the other so that their sides coincide.

64. Two equal angles in the same plane may always be brought into coincidence by applying a slide followed by a rotation (§ 35). Let the student explain.

65. Unequal angles. Sum and difference of two angles.

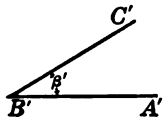


FIG. 20.

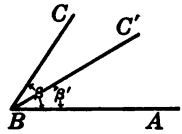


FIG. 21.

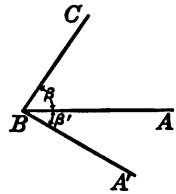


FIG. 22.

Given two angles ABC and $A'B'C'$ (Figs. 20, 21). Imagine angle $A'B'C'$ to slide until the vertex B' coincides with the vertex B, and then to rotate about B as a center until one of the sides $B'A'$ of angle $A'B'C'$ coincides with BA, while $B'C'$ and BC are on the same side of BA. Then if the side $B'C'$ of angle $A'B'C'$ is between the sides BA and BC (Fig. 21), angle ABC is said to be *greater* than angle $A'B'C'$. Also angle $C'BC$ is called the *difference* of angles ABC and $A'B'C'$. In symbols, $\angle ABC > \angle A'B'C'$; $\angle C'BC = \angle ABC - \angle A'B'C'$. If after placing the angle $A'B'C'$ in this position, BC is between the sides of angle $A'B'C'$, angle ABC is said to be *less* than angle $A'B'C'$. In symbols $\angle ABC < \angle A'B'C'$. If after the rotation the common side is between the other two sides (Fig. 22), the angle formed by these sides is called the *sum* of angles ABC and $A'B'C'$, $\angle A'BC = \angle ABC + \angle A'B'C'$.

66. The right angle. If the sides of an angle are perpendicular, the angle is called a *right angle* (R). Hence, *two perpendicular lines form four right angles.*

The fact that each of two perpendicular lines is symmetric with respect to the other (§ 48) leads at once to the following:

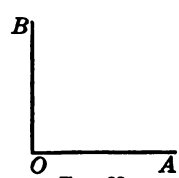


FIG. 23.

67. *All right angles are congruent.*

68. We may now infer the following: In a plane *through a given point on a straight line one, and only one, perpendicular can be drawn to that line.* (Cf. § 57.)

69. Practical units. For practical purposes angles are measured in *degrees, minutes, seconds.* A right angle contains 90 degrees, a degree 60 minutes, and a minute 60 seconds.

70. Draftsmen's triangles. For drawing perpendicular lines, for constructing angles of certain magnitudes, and for other uses to be explained later, practical draftsmen employ a pair of *triangles.* These are instruments having three straight edges forming three angles, the edges of the one, called the *45-degree triangle* (Fig. 24), form angles of 90° and 45°, respectively; the edges of the other, called the *60-degree — 30-degree triangle* (Fig. 25), form angles of 90°, 60°, and 30°, respectively.

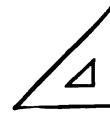


FIG. 24.

FIG. 25.

71. *An angle AOB has an axis of symmetry,* for evidently the angle may be turned over so that the sides *OA* and *OB* change places (§ 37). Since the point *O* remains fixed, the axis must pass through the vertex. *An angle cannot have more than one axis of symmetry.* Why?

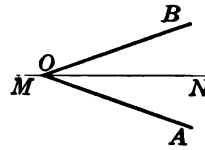


FIG. 26.

72. The bisector of an angle. The axis of symmetry of an angle is called the *bisector* of the angle; and the axis is said to *bisect* the angle.

73. PROBLEM. *To bisect an angle AOB.*

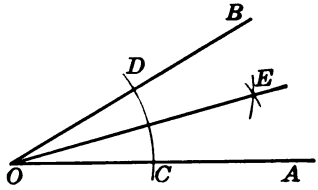


FIG. 27.

Construction. Since *O* is on the axis, the problem will be solved if we can obtain another point on the axis. Why? To

do this choose any convenient radius and describe a circle with center O, cutting OA in C and OB in D. From centers C and D and the same radius sufficiently long describe arcs intersecting in E. Draw OE. OE is the required line. (Let the student prove this. See also Ex. 7, § 49.)

74. We may now prove the following important proposition: *Every point on the bisector of an angle is equally distant from the sides of the angle.*

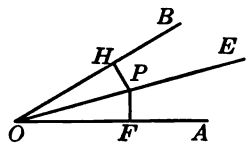


FIG. 28.

In the figure let OE be the bisector of angle AOB, and let P be any point on OE. The distance of P from a side of angle AOB is the length of the perpendicular drawn from P to the side. We are required to prove that the perpendiculars PF and PH are equal. To do this imagine the figure to turn over on OE. Then P will remain fixed and OA and OB will exchange positions. Therefore, PF and PH must also exchange places (§ 53). Therefore, $PF = PH$ (§ 9).

75. **EXERCISES**

1. By the use of compasses and straightedge construct angles of 90°, 45°, 22½°, 135°.

2. Given an angle AOB, and a straight line cutting the sides OA and OB in C and D, respectively. Find a point on CD equally distant from OA and OB.

3. AB and CD are diameters of a circle with center O. Draw a diameter bisecting angle AOC. Prove that this diameter also bisects angle BOD, and that each of its extremities is equally distant from A and C.

4. A and B are two points on a circle. Find two points on the circle, each equally distant from A and B.

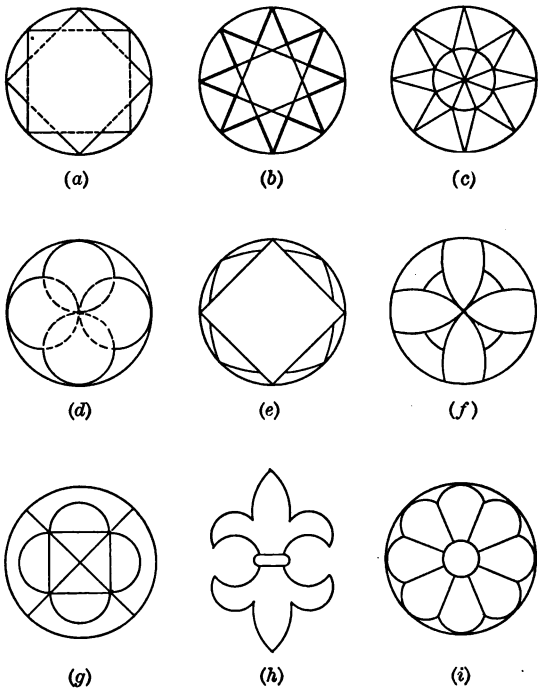
5. Divide a circle into eight parts so that the chords joining successive points are equal.

6. Through any point on the bisector of any angle AOB draw a perpendicular to the bisector, cutting OA in C and OB in D. Prove that DC is bisected by OP, that OC is equal to OD, and that angle OCD is equal to angle ODC.

7. On the sides of an angle AOB take points C and D equally distant from O. Draw CD. Prove that angle OCD is equal to angle ODC.

8. Show that the lines drawn from the extremities of a line segment to any point on the perpendicular bisector of the segment make equal angles with the perpendicular and with the segment.

9. In the following figures are shown the geometric foundations for various designs used in window tracery. Many variations may be obtained by coloring certain portions of the drawings, and by other artistic devices. Let the student draw the designs, adding or inventing embellishments if desired.



PARALLEL LINES

76. Parallel lines. Two (unlimited) straight lines in the same plane which do not meet are said to be *parallel* (∥).

77. *Two straight lines in the same plane which are perpendicular to the same straight line are parallel.*

For, if they are not parallel they will meet. But no more than one perpendicular can be drawn through a given external point to a given straight line (§ 53).

78. FUNDAMENTAL PROPOSITION. *Through a given point not on a given straight line one straight line, and only one, can be drawn parallel to the given line.*

79. PROBLEM. *Through a given point P not on a given straight line AB, to draw a line parallel to that line.*

(Let the student make the construction by the use of compasses and straightedge.)

80. For drawing parallel lines practical draftsmen use a *T-square* in connection with a *drafting board* (see Fig. 1, p. 1) or a *draftsman's triangle* (see Fig. 29).

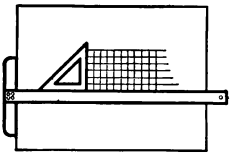


FIG. 29.

81. *Two straight lines which are parallel to the same straight line are parallel to each other.*

For, if they are not parallel, they will meet; but two intersecting lines cannot both be parallel to the same straight line (§ 78).

82. *A straight line which is perpendicular to one of two parallel lines is perpendicular to the other also.*

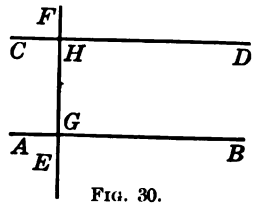


FIG. 30.

For, let AB and CD be two parallel lines, and let EF be perpendicular to AB, cutting AB and CD in G and H, respectively. A perpendicular to EF at H is parallel to AB, and coincides with CD. Why?

EXERCISE: Under what conditions is Ex. 1, § 59, impossible?

83. *Two straight lines in the same plane which are perpendicular respectively to two intersecting lines must meet.*

The lines cannot be parallel, for then each would be perpendicular to both of the given intersecting lines (§ 53).

84. *Two parallel lines are everywhere equally distant.*

For, two parallel lines are symmetric with respect to any perpendicular (§ 50).

85. EXERCISES

1. Two straight streets each 60 feet wide intersect at an angle of 45°. The building lines are 20 feet from the curbing. Draw a plan of the corner.

2. Two straight streets each 60 feet wide intersect at an angle of 135°. Draw a plan showing how to locate the corner stone of a building 20 feet from one street and 30 feet from the other.

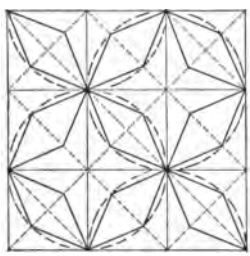
3. A straight railway passes two miles from a town. A place is four miles from the town and one mile from the railway. Find by construction the places that answer this description.

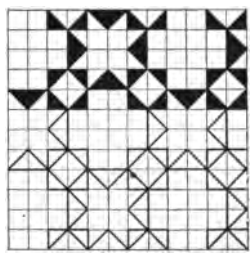
4. A watch dog is tethered to a stretched wire 30 feet long by a rope. The rope is 15 feet long and is fastened to the wire by a ring free to move along the whole length of the wire. Draw a plan showing the dog's complete range, assuming that the rope when stretched is horizontal.

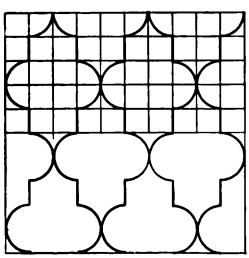
5. A horse is tethered to one corner of a house 30 feet wide by 60 feet long, by a rope 100 feet long. Draw a plan showing the horse's complete range.

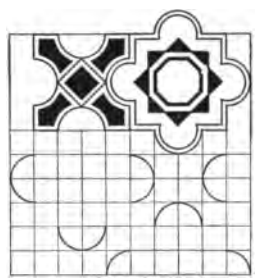
6. A and B are two points equally distant from and on the same side of a straight line CD. Show that the straight line determined by A, B is parallel to CD.

7. The following figures are examples of pattern designs for mosaics, tiles, floorcloths, etc. The constructions are based on "the square net," that is, a series of parallel lines equal distances apart, crossed by a like series of parallels at right angles to the first.









MORE ABOUT THE CIRCLE. TANGENTS

86. In § 44 it was proved that a circle is symmetric with respect to any diameter. As an immediate consequence of this proposition we infer the following:

I. *The perpendicular drawn to a chord through the center of a circle bisects the chord and the arcs subtended (cut off) by the chord.*

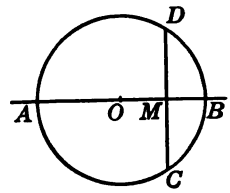


FIG. 31.

87. II. *The straight line drawn through the center of a circle and bisecting a chord* (which is not a diameter) *is perpendicular to the chord.*

88. III. *The perpendicular drawn to a chord through its midpoint passes through the center of the circle.*

89. PROBLEM. *To describe a circle through three given points A, B, and C not on a straight line.*

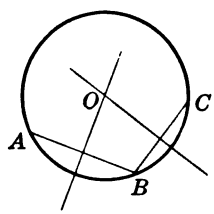


FIG. 32.

Construction. If A, B, C are on a circle, the segments AB and BC are chords. Reference to the preceding paragraph shows that the problem will be solved by drawing the perpendicular bisectors of AB and BC. These lines meet in a point which is equally distant from A, B, and C. Let the pupil explain. See §§ 83, 58.

90. Relative positions of a straight line and circle in the same plane. A straight line and a circle in the same plane may have three different relative positions, as follows:

I. They may have no point in common (Fig. 33), in which case the straight line lies wholly without the circle; and its distance from the center, that is the perpendicular segment OP from the center to the line, is greater than a radius (§ 25).

II. They may have two points, A and B, in common (Fig. 34), in which case the perpendicular segment OP is less than a radius (§ 54).

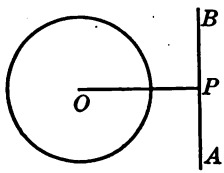


FIG. 33.

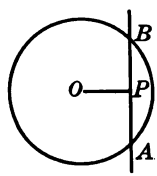


FIG. 34.

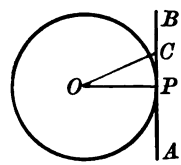


FIG. 35.

III. Finally, if the line is so situated that the distance OP from the center to the line is equal to the radius, the point P is on the circle (Fig. 35). Moreover, it is the only point which is common to the straight line and circle; for the distance from any other point C on AB to O is greater than the radius OP (§ 54). Therefore, the point C and hence any other point on AB except P lies outside the circle (§ 25).

91. A tangent to a circle. A straight line in the plane of a circle which has only one point in common with the circle is said to be *tangent* to the circle. The circle is also said to be *tangent* to the line. The common point is called the *point of contact*, or the *point of tangency*.

92. The result of § 90, III may now be stated as follows: *A straight line in the plane of a circle and perpendicular to a radius at its outer extremity is tangent to the circle.*

93. *A straight line which is tangent to a circle is perpendicular to the radius drawn to the point of contact.* For the radius is shorter than the segment joining the center of the circle to any other point on the tangent (§ 25, § 54).

Since in a given plane only one perpendicular can be drawn through a given point to a given straight line, we infer the following as immediate consequences of the preceding proposition:

94. I. *The perpendicular to a tangent, drawn through the point of contact, passes through the center of the circle.*

95. II. *The perpendicular to a tangent, drawn through the center of the circle, passes through the point of contact.*

96. Problem. *Through a given point to draw a tangent to a given circle.*

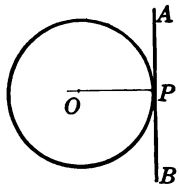


Fig. 36.

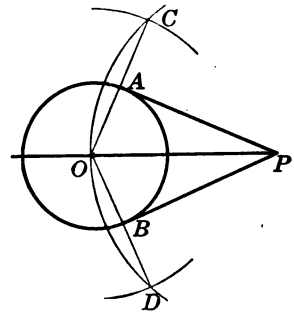


Fig. 37.

Construction. Let O be the center of the given circle, and let P be the given point.

I. If P is on the circle (Fig. 36), we have merely to draw the radius OP, and draw AB perpendicular to OP through P. Then AB is the line required. Why?

II. If P is outside the circle (Fig. 37), the solution is not so simple. In this case we must find a point A on the circle such that PA will be perpendicular to OA. Reference to § 87 shows that the problem will be solved if we can make P the center of a circle in which A is the mid-point of a chord having one extremity at O. Therefore, the construction proceeds as follows: With P as a center describe a circle through O. Then with O as a center and a radius equal to twice the radius of the given circle describe a circle intersecting the first in C and D. Join C and D to O, intersecting the given circle in A and B. Draw PA and PB. Either of these lines is the required tangent. Let the student explain why.

97. A tangent from an external point to a circle is the segment of the tangent between the external point and the point of contact.

NOTE. It will be noted that here the word "tangent" is used to denote a line segment, where previously (§ 91) it was used to denote a whole line. The context will always make clear which meaning is to be attached to the term.

98. *The tangents drawn to a circle from the same external point are equal and make equal angles with the line joining the point to the center of the circle.*

For, referring to Fig. 37, since the circles with centers O and P are symmetric with respect to OP, if the figure is turned over on OP, OC and OD will exchange places, consequently A and B, the mid-points of OC and OD, will exchange places. Therefore $PA = PB$, and $\angle OPA = \angle OPB$.

99. Concentric and excentric circles. Circles which are in the same plane and have the same center are called *concentric* circles; if they have different centers, they are called *excentric* circles.

100. Tangent circles. If two circles in the same plane are tangent to the same straight line at the same point, they are said to be *tangent* to each other. The common point is called the

point of contact, or the *point of tangency*. The circles are said to be tangent *externally* (Fig. 38), or *internally* (Fig. 39) according as they lie on opposite sides, or on the same side of this common tangent.

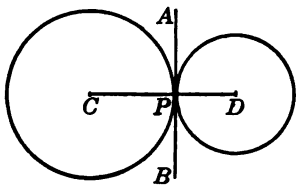


FIG. 38.

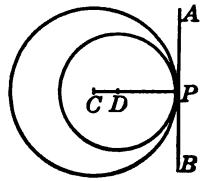


FIG. 39.

101. *The line of centers of two tangent circles passes through their point of contact.* For the perpendicular drawn to their common tangent in the plane of the circles and passing through the point of contact also passes through the centers of the circles (§ 94, § 3).

102. **EXERCISES**

1. To find the center of a circle, having given an arc. Apply § 88.

2. To bisect an arc of a circle (that is, divide it into two congruent parts) (*a*) when the center of the circle is accessible; (*b*) when the center is not accessible.

3. Given a circle with center O, and a straight line AB. To draw a tangent to the circle (*a*) which shall be parallel to AB; (*b*) which shall be perpendicular to AB.

HINT: Draw the perpendicular to AB through O.

4. Given an arc of a circle and a point P on the arc. To draw a tangent to the arc through the point P, when the center of the circle is inaccessible.

HINT: From P as a center describe a circle with any convenient radius cutting the arc in A and B. P is the mid-point of the arc AB.

5. Given a straight line AB, a point P on AB and a point Q not on AB. To describe a circle tangent to AB at P and passing through Q.

HINT: If the circle passes through P and Q, the segment PQ is a chord. Apply § 88, § 94.

6. Two street curbings which intersect at an angle of $67\frac{1}{2}°$ are joined by an arc of a circle having a radius of 15 feet, and tangent to the two curbings. Draw a plan of the corner.

7. Prove that parallel lines which meet a circle cut off congruent arcs between the lines.

SUGGESTION: Draw a diameter perpendicular to the lines.

8. Draw the tangents to a circle from a point outside the circle. Prove that the chord joining the points of contact is perpendicular to and is bisected by the line joining the point to the center of the circle; also that the chord makes equal angles with the tangents.

9. Given two circles tangent at a point C, either externally or internally. Let AB be the common tangent drawn through the point C. Prove that the tangents drawn from any point P on AB to the given circles are equal. Apply § 98.

10. Given a circle, a point A on the circle, and a segment r. To describe a circle tangent to the given circle at A and having a radius equal to r. See § 101.

11. Given a circle, a point A on the circle, and a point B outside the circle. To describe a circle passing through B and tangent to the given circle at A.

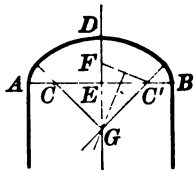
HINT: If the required circle passes through A and B, the segment AB is a chord. See §§ 88, 101.

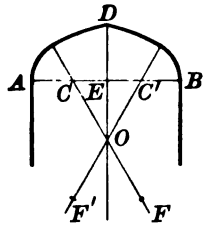
12. Given two circles with centers C and D, and radii p and r, respectively, and a segment s. To describe a circle tangent to the given circles and having a radius equal to s.

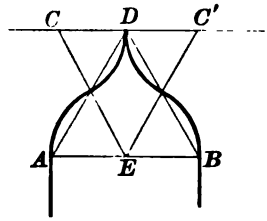
SUGGESTION: Describe circles from centers C and D and radii equal to $p + s$ and $r + s$, respectively.

13. Given three segments p, r, s. To describe three circles tangent to each other and having radii equal to p, r, and s, respectively.

14. A branch of an interurban railway meets the main line at an angle of 45°. The branch is connected with the main line, on one side of the latter, by circular arcs, beginning at a point on the branch 200 feet from the crossing. Draw a plan of the crossing.



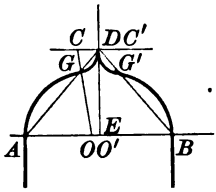




15. The left-hand figure above is a drawing of a three-centered arch. To construct the arch, the span AB, the height ED, and C, C', the centers of the small arcs, are given. Also $AC = C'B = FD$.

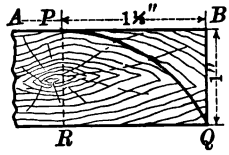
16. The middle figure above is a drawing of the depressed Tudor arch described from four centers. To construct the arch the span AB is given ; also $AC = C'B = \frac{1}{4} AB$, and $CC' = CO = OF$.

17. The right-hand figure above is a drawing of the ogee arch described from three centers. To construct the arch. the span AB is given. All arcs are described with the same radius.



18. The adjacent figure is a drawing of the ogee arch described from four centers. To construct the arch, the span AB and the height ED are given. Also, the point G is chosen arbitrarily on AD. The centers C, C' are on a line through D parallel to AB.

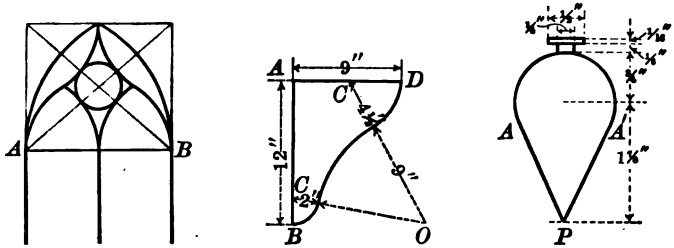
19. The adjacent figure shows the cross section of a type of "molding" cut from a rectangular piece of timber. The problem is to describe the arc PQ, so that it will be tangent to AB. In the language of carpentry this is called "ramping the molding to a given point." Show how to lay out the cross section.



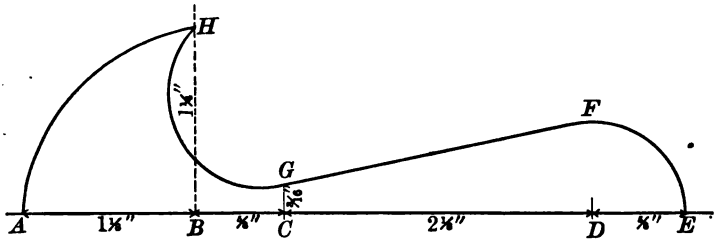
20. The left-hand figure at top of page 33 shows simple leading lines for the window tracery of an equilateral arch. Copy the design, given the span AB.

21. The middle figure shows a plan of a wall-bracket support. Copy the drawing to convenient scale.

22. The right-hand figure shows a profile of a "plummet" or "plumb bob." Make a full-size drawing of the figure. Observe that PA and PA' are tangents.



23. The following figure is a plan for a "darning hemisphere." Make a full-size drawing of the plan. Observe that GF is tangent to the arcs EF



and GH; also that the center of the circle AH is on AE.

HINT: Draw the arcs EF and AH, first, then the tangent GH, and finally the arc GH.

COMMON TANGENTS

103. A common tangent to two circles. A straight line which is tangent to two circles is called a *common tangent* to the circles. The line is called an *exterior* tangent (Fig. 40) or an *interior* tangent (Fig. 41), according as the circles lie on the same side or on opposite sides of the line.

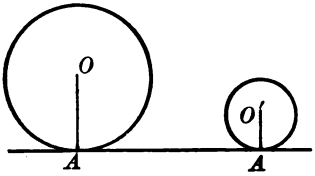


FIG. 40.

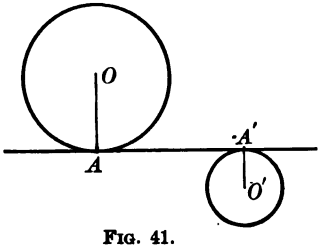


FIG. 41.

104. Problem. *To draw a common exterior tangent to two given circles.*

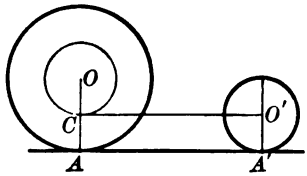


Fig. 42.

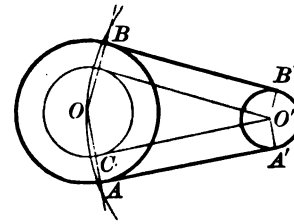


Fig. 43.

Construction. In order to discover how to make this construction consider two unequal circles with centers O and O' (Fig. 42) drawn tangent to the straight line AA' at A and A', respectively, and lying on the same side of AA'. Also let the radius OA be greater than the radius $O'A'$. Draw $O'C$ parallel to $A'A$, intersecting OA in C. With center O and radius OC describe a circle. Call this the auxiliary circle. Then $O'C$ is tangent to the auxiliary circle. For $O'C$, being parallel to $A'A$, is perpendicular to OC (§§ 82, 92). Moreover, since the segment AC is equal to the radius $A'O'$ (§ 84), OC, the radius of the auxiliary circle, is equal to the difference of the radii of the given circles. Therefore, by reversing the order of construction an exterior tangent to two given circles with centers O and O' (Fig. 43) may be drawn as follows:

Let the circle with the center O be the greater. From O as a center with a radius equal to the difference of the radii of the given circles, describe an auxiliary circle. From O' draw a straight line tangent to the auxiliary circle at C (§ 96, II). Draw the radii OA and $O'A'$ perpendicular to $O'C$. Then A and A' are the points of contact of the required tangent. In like manner the common exterior tangent BB' may be drawn.

105. In order that the above construction may be possible it must be possible to draw tangents from O' to the auxiliary circle. This is only possible if O' is outside this circle. Hence, if either circle lies within the other, the construction is impossible.

If the circles have equal radii, the construction is greatly simplified. Let the student explain.

106. PROBLEM. *To draw a common interior tangent to two given circles.*

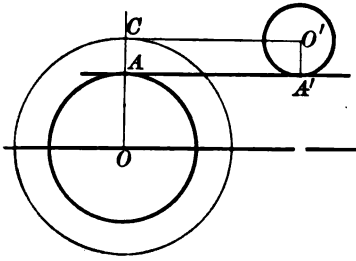


FIG. 44.

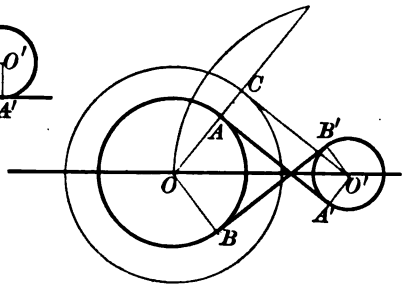


FIG. 45.

Construction. In order to discover how to make this construction, consider two circles with centers O and O' (Fig. 44) drawn tangent to the straight line AA' at A and A', respectively, and lying on opposite sides of AA'. Draw $O'C$ parallel to $A'A$, intersecting OA produced in C. With center O and radius OC describe an auxiliary circle. $O'C$ is tangent to this circle. Why? Moreover, OC, the radius of the auxiliary circle, is equal to the sum of the radii of the given circles. Why? Therefore, by reversing the order of construction an interior tangent to two given circles with centers O and O' (Fig. 45) may be drawn. Let the student explain.

Query. Under what conditions would the construction be impossible?

107. EXERCISES

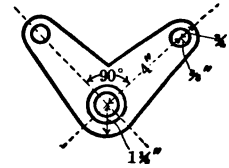
1. Draw two circles (1) which have two exterior tangents and only one interior tangent; (2) which have two exterior tangents and no interior tangent; (3) which have one common tangent; (4) which have no common tangent.

2. If two circles are tangent externally, prove that the common interior tangent bisects the exterior tangents. HINT: Apply § 98.

3. Given two unequal circles which are exterior to each other (and not tangent). Prove that the exterior tangents and the interior tangents, respectively, meet on the line drawn through the centers of the circles. Also prove that the segments of the common tangents between the points of contact are respectively equal.

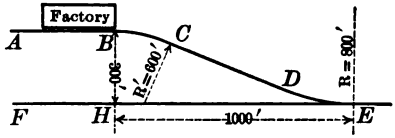
HINT: Show that the figure is symmetrical with respect to the line drawn through the centers of the circles.

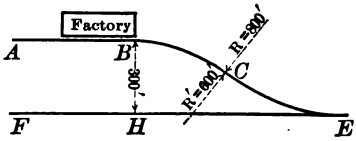
4. A belt runs over two pulleys whose diameters are 3 in. and 2 in., respectively, and whose centers are 8 in. apart. Make drawings for the two cases when the belt does not cross and when it does cross.



5. The adjacent figure shows a simple pattern of a valve lever. Make a drawing of the pattern to scale according to the dimensions given.

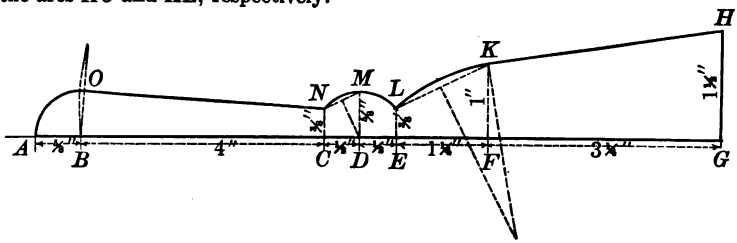
6. The adjacent figure is a plan of a railway siding. *AB* is the siding parallel to the main line *FE* and 300 feet distant. *BC* and *ED* are arcs of circles tangent to *AB* and *FE*, respectively, and joined by the common tangent *CD*. Draw the plan to scale according to the dimensions given.





7. Suppose in Ex. 6 it is desired to connect the arcs *BC* and *DE* directly without the aid of the tangent *CD*, the distance *HE* being undetermined. Make a drawing of the plan.

8. The figure below is a drawing for a "carver's mallet." Reproduce the drawing using a large scale. Observe that *ON* and *HK* are tangent to the arcs *AO* and *KL*, respectively.



ANGLE AND CIRCLE

108. Central angle. An angle which is formed by two radii of a circle produced is called a *central angle.* The angle is said to *intercept* the arc of the circle included between its sides.

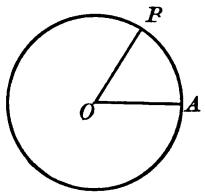


FIG. 46.

109. Arcs subtended by chords. An arc of a circle is said to be *subtended* by the chord joining its end points.

110. Equal arcs. Two arcs of circles which are congruent are said to be *equal.*

111. Major and minor arcs. Any chord AB of a circle, other than a diameter, evidently divides the circle into two unequal arcs, one greater and one less than a semicircle. These are called *major* and *minor* arcs, respectively. Unless referred to explicitly as the major arc, "the arc subtended by the chord AB," or simply "the arc AB," refers to the minor arc subtended by the chord.

112. *Equal arcs of congruent circles in the same plane may be brought into coincidence by a slide followed by a rotation.*

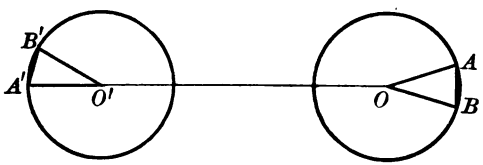


FIG. 47.

For, suppose AB and $A'B'$ are equal arcs of the congruent circles with centers O and O'. Imagine the circle with center O' to slide along the line $O'O$ until O' coincides with O. Then the circles will coincide, and the arc $A'B'$ may be brought into coincidence with arc AB by rotating the circle about O, since all points of the circles are the same distance from the center O.

113. It follows immediately, therefore, that *equal arcs in the same circle or in congruent circles are subtended by equal chords, and are also intercepted by equal central angles.* For when the arcs are brought into coincidence, the chords and the central angles will also coincide, respectively.

114. Similarly, if two central angles are equal, they may be made to coincide. We may therefore infer immediately the following: *In the same circle or in congruent circles equal central angles intercept equal arcs and are subtended by equal chords.*

115. We may now prove the following important proposition: *In the same circle or in congruent circles equal chords subtend equal arcs and equal central angles.*

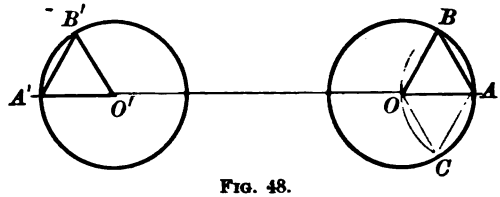


FIG. 48.

In the congruent circles with centers O and O' let chord AB be equal to the chord $A'B'$. We wish to prove that arc AB is equal to arc $A'B'$, and that angle AOB is equal to angle $A'O'B'$. Imagine the circle with center O' to slide along the line $O'O$ until O' coincides with O, and then to rotate about O until the arc $A'B'$ takes the position AC, A being between B and C. Chord AC is then equal to chord AB. Why? Therefore a circle with center A and radius AB will pass through C. Therefore from the symmetry of two intersecting circles, arc AB is equal to arcs AC or $A'B'$; and angle AOB is equal to angle AOC or its equal, angle $A'O'B'$.

116. The measure of an angle. For practical purposes a circle is conceived to be divided into 360 equal arcs called *degrees* (degrees of arc). A degree contains 60 *minutes*, and a minute contains 60 *seconds*.

Since equal arcs of a circle are intercepted by equal central angles (§ 114), a unit angle at the center intercepts a unit arc on the circle. In this sense, *a central angle is said to be measured by its intercepted arc.*

117. PROBLEM. *On a given half line $O'A'$ to construct an angle equal to a given angle AOB.*

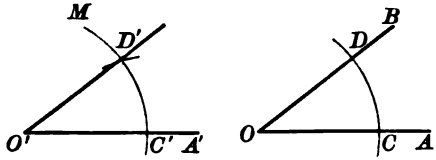


FIG. 49.

Construction. From O and O' as centers and any convenient radius describe circles intersecting OA, OB, and $O'A'$ in C, D, and C', respectively. Then with center C' and a radius equal to the chord CD describe a circle intersecting the circle $C'M$ in D'. Draw $O'D'$. Angle $C'O'D'$ is the required angle. Why?

118. The protractor. The protractor (Fig. 50) is an instrument used by draftsmen for measuring angles. Its construction depends on the division of a circle into 360 equal arcs.

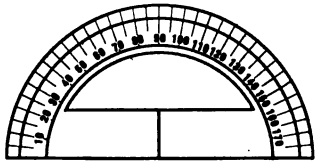
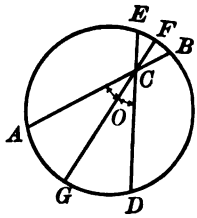


FIG. 50.

NOTE. There is no general geometric method for dividing an arc of a circle into any number of equal arcs. An approximate division of an arc may be obtained by means of the dividers as follows: Estimate the length of a chord which subtends an arc of the required magnitude. Test this estimate by trial, correcting it if necessary, and repeat until the required accuracy is obtained.

119. EXERCISES

1. Construct an angle equal to twice a given angle.
2. Construct angles equal to the sum and difference of two given angles.
3. Prove: In the same circle or in congruent circles equal chords are equally distant from the center.

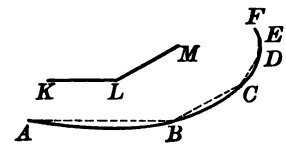


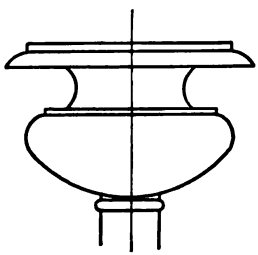
4. Prove: If two intersecting chords of a circle make equal angles with the diameter drawn through the point of intersection, the two chords are equal.

HINT: Imagine the adjacent figure turned over on the axis FG.

5. Prove: Two chords of a circle drawn perpendicular to a third chord at its extremities are equal.

6. In constructing ornamental designs architects and artists sometimes employ a curve which has been called the *infinite curve*. Its geometric basis is as follows: A broken line $ABCDE \cdots$ is constructed such that the angles ABC, BCD, $CDE \cdots$ are each made equal to a given angle KLM, and the lengths of the segments AB, BC, CD, $DE \cdots$ are determined by a given law. In the adjacent figure each segment beginning with BC is one half the preceding one. A smooth curve is then drawn through the points $ABCDE \cdots$ freehand. By varying the law determining the lengths of the segments, and the size of the angle KLM, a variety of graceful curves may be constructed.



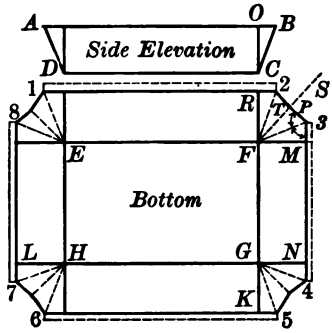


7. The adjacent figure shows a design for a balluster based on the infinite curve. Make a drawing of the design, constructing the curves on one side of the figure by the method explained in Ex. 7; then obtain the curves on the other side by the principles of symmetry.

8. Design the profile for a vase, a bowl, a pitcher, or other simple ornamental object by the use of the infinite curve.

9. The figure (on next page) shows a plan of a roasting pan with beveled sides. The special feature of the pan is that the corners are folded to give the required bevel and at the same time have the folded metal come directly under the wired edge of the pan. A pan folded in this way gives a watertight joint without any soldering.

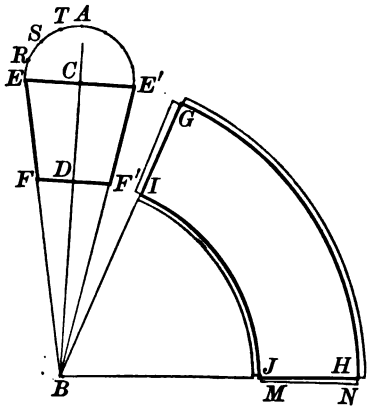
To construct the plan determine the rectangular bottom $EFGH$ and the



side elevation $ABCD$. Take FM, LH, GK, etc., each equal to CB. Also take $3M$, $N4$, $L7$, etc., each equal to OB. Draw FS bisecting $\angle RFM$; construct $\angle T3F = \angle F3M$; let $T3$ cut FS in P and join P to 2. Transfer this cut to each of the other corners. Dotted lines indicate wire allowance.

If convenient, draw the plan on cheap manila paper, and test by cutting out the pattern and forming up the model.

10. The figure below shows a plan of a flaring bucket. To construct the plan, determine the depth CD and the diameters of the top and bottom EE' and FF'. Produce EF and $E'F'$ to meet at B. From B as a center and radii BF and BE describe arcs of circles IM and GN, respectively. Describe the semicircle EAE' and divide it into a number of equal arcs, ER, RS, etc. Then construct on the arc GN a broken line consisting of twice as many chords, each equal to the chord ER. Join the extremities of the arc GH to B. Then the figure $GHJI$ is the required plan.



If convenient, draw the plan on cheap manila paper, and test by cutting out the pattern and forming up the model.

It should be noted that this is a practical plan based on approximations. When the pattern is formed up into shape, the arcs GH and IJ will not be geometrically correct, but they will be close approximations to the circles with diameters EE' and FF'.

ADJACENT ANGLES; VERTICAL ANGLES

120. Adjacent angles. Two angles ABC and ABD with a common vertex B and a common side AB, such that the other two sides BC and BD (called *exterior sides*) are on opposite sides of the common side AB, are said to be *adjacent.*

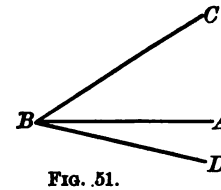


FIG. 51.

121. *If two adjacent angles DBA and ABC have their exterior sides in a straight line, their sum is equal to two right angles* ($2R$).

For, draw BE perpendicular to CD. Then $\angle DBA + \angle ABC = \angle DBE + \angle EBC = 2R$.

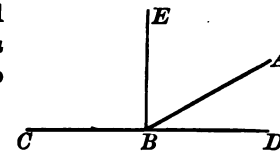


FIG. 52.

122. From this we infer immediately the following: *When one straight line meets another straight line and makes two adjacent angles equal, each is a right angle, and the lines are perpendicular.*

123. Also: *If two or more straight lines in a plane pass through a common point,*

I. *the sum of the consecutive adjacent angles formed on one side of any line drawn through the point is two right angles* ($2R$);

II. *the sum of all the consecutive adjacent angles formed is four right angles* ($4R$).

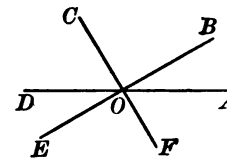


FIG. 53.

124. Complementary angles. Two angles α and β whose sum is a right angle are said to be *complementary*, and each is called the *complement* of the other. Hence, if α, β are complementary,

$$\alpha + \beta = R.$$

125. *Complements of the same angle, or of equal angles, are equal.*

For, let α and α' be complements of the same angle or of equal angles. Then $R - \alpha$ and $R - \alpha'$ are equal. But from

$$R - \alpha = R - \alpha',$$

follows

$$\alpha = \alpha'.$$

126. Supplementary angles. Two angles α and β whose sum is two right angles are said to be *supplementary*, and each is called the *supplement* of the other. Hence, if α, β are supplementary,

$$\alpha + \beta = 2\,R.$$

127. *Supplements of the same angle or of equal angles are equal.*

For, let α and α' be supplements of the same angle or of equal angles. Then $2\,R - \alpha$ and $2\,R - \alpha'$ are equal. But from

$$2\,R - \alpha = 2\,R - \alpha',$$

follows

$$\alpha = \alpha'.$$

128. *If two adjacent angles ABC and CBD are supplementary, their exterior sides BA and BD are in a straight line.*

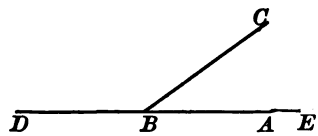


FIG. 54.

For, suppose DB produced to E. Then $\angle\, EBC$ is supplementary to $\angle\, CBD$ (§ 121). Therefore,

$$\angle\, ABC = \angle\, EBC \text{ (§ 127)},$$

and BA coincides with BE, that is, with DB produced.

129. Vertical angles. Two angles AOC and BOD having the same vertex O, and such that the sides of the one are the prolongations of the sides of the other, are called *vertical angles.*

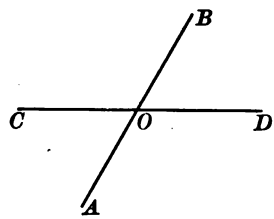


FIG. 55.

130. *Vertical angles are equal.* Why?

130 a. Acute and obtuse angles. An angle (less than two right angles) is said to be *acute* or *obtuse* according as it is less than or greater than a right angle.

131. **EXERCISES**

1. If one of the angles formed by two intersecting lines is 45°, what are the other angles?

2. If one of the angles formed by two intersecting lines is denoted by α, express the other angles.

3. What is the complement of 30° ? of an acute angle α ?

4. What is the supplement of 30° ? of an angle α ?

5. What angle is equal to its supplement ?

6. Find the angle that is five times its supplement.

7. Find the angle whose supplement is three times its complement.

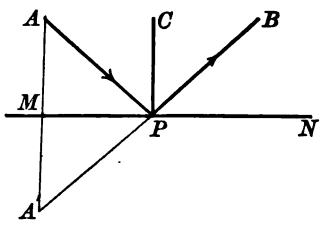
8. If two adjacent angles are supplementary, prove that their bisectors are perpendicular.

9. Prove that the straight line which bisects one of two vertical angles bisects the other also.

HINT: Turn the figure on the bisector as an axis.

10. To draw a line through the vertex of a given angle and forming equal angles with the sides.

11. A ray of light having its origin at a point A and striking a plane mirror at P is reflected along a straight line PB such that AP and PB make equal angles in the same plane with a perpendicular PC to a certain line MN on the mirror. This is expressed in physics by saying that *the angle of reflexion* CPB is equal to *the angle of incidence* APC.



To determine the path of a ray of light having its origin at a given point A, and which is reflected from the given straight line MN to the given point B. See Ex. 7, § 59.

CENTRAL SYMMETRY

132. Symmetry with respect to a point. Imagine Fig. 56 to rotate in the plane of the figure about O, the mid-point of MN, until M and N change places. Let Fig. 57 be a duplicate of Fig. 56 together with the position of the latter after the rotation. Then Fig. 57 is symmetric with respect to O as a center of symmetry.

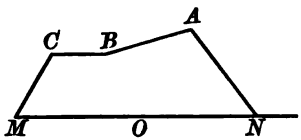


FIG. 56.

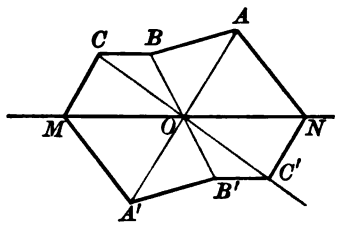


FIG. 57.

In general, a figure is said to be *symmetric with respect to a point as a center of symmetry* if, after the figure is rotated in its plane about that point through two right angles, the figure coincides point for point with its original position.

133. Corresponding parts. Any two parts of the figure (points, lines, angles, etc.) which simply change places when the figure is thus rotated are called *corresponding parts.*

As immediate consequences of the preceding paragraphs we have: *If a figure is symmetric with respect to a point,*

134. I. *Corresponding parts of the figure are congruent.*

135. II. *A point which corresponds to itself is the center of symmetry.*

136. III. *A straight line which corresponds to itself passes through the center of symmetry.*

137. IV. *Segments joining corresponding points pass through the center of symmetry and are bisected by it.*

138. *A straight line is symmetric with respect to any one of its points.*

139. *A circle is symmetric with respect to its center.*

140. *The figure formed by two straight lines which cut each other is symmetric with respect to their common point.*

Let the student prove this.

141. **EXERCISES**

1. Which letters of the alphabet have a center of symmetry?

2. Given three points A, B, and C. Find points A' and B' in a symmetry with respect to C as a center.

3. Arrange two points and a straight line in a symmetry with respect to a center.

4. Show how to bisect a line segment by the use of a draftsman's triangle or set square alone.

5. Given two equal circles tangent at a point P. Prove that any line segment having its extremities on the circles and passing through P is bisected by the latter.

6. Given a straight line AB and a point O not on AB. Construct a line $A'B'$ corresponding to AB in a symmetry with respect to O as a center.

7. Discover a method for measuring the distance between two points on level ground, when the surface between the points is obstructed by some object, as a house.

CHAPTER II

PARALLEL LINES

FORMAL STATEMENT OF PROOF

142. The student should now understand, in a general way, the nature of a geometric investigation. During the study of Chapter I he has observed that one object of geometry is to investigate the properties of geometric figures with the view of discovering how to make certain constructions; he has also observed that the method of investigation is a process of reasoning wholly independent of physical measurement. In Chapter II we begin a more systematic study of geometry. It becomes necessary, therefore, to define the following *logical terms:*

143. A proposition. A *geometric proposition* is a statement asserting a property of a geometric figure, or figures.

144. A geometric proof. To *prove* a proposition, or a construction, is to show that it follows as a consequence of definitions, propositions, and constructions previously proved or admitted. Such a process is called a *demonstration*, or a *proof.*

145. A fundamental proposition. It is manifestly impossible to prove all propositions. In order to provide a basis for demonstrations, some propositions must be assumed, or admitted, without proof. Such an assumed proposition is called a *fundamental proposition*, or a *postulate.* Thus, "One straight line, and only one, can be drawn through two given points" (§ 3) is, in our treatment of geometry, a fundamental proposition.

146. A theorem. A proposition which may be proved on the basis of the fundamental propositions and definitions is called a *theorem.*

Thus, "Every point on the bisector of an angle is equally distant from the sides of the angle" (§ 74) is a theorem.

147. A corollary. A theorem which is closely connected with a proposition or a definition and which is easily deduced therefrom is called a *corollary.*

Thus, "Two straight lines can intersect in no more than one point" is a corollary of the fundamental property of the straight line referred to in § 145; and "All radii and all diameters of a circle are respectively equal" is a corollary of the definition of a circle (§§ 24, 18, 20).

148. Qualifications of a definition. Since the definitions are always an important factor in a logical discussion, it is important that we should call attention to the following properties of any good definition:

(1) A definition should affirm a *sufficient* number of properties of the thing defined to determine it completely, and *no more.*
(2) The properties affirmed must not contradict previously established propositions.

EXAMPLES: "A straight line in the plane of a circle and meeting it in only one point is said to be tangent to the circle" is a correct definition of a tangent to a circle; but "a straight line which meets a circle in only one point and is perpendicular to the radius drawn to the point of contact is called a tangent" is not a correct definition of a "tangent" because it is both insufficient and redundant. It is insufficient because the line and circle must be in the *same plane.* Why is it redundant?

149. Undefined terms. It is, of course, impossible to define all terms; a knowledge of some terms must be admitted as a basis for defining other terms. Thus, the terms point, straight line, plane, distance, etc., have been left undefined in this text, a knowledge of the meaning of these terms being assumed.

150. Hypothesis and conclusion. A theorem consists of two parts: the *hypothesis,* or that which is assumed or given; and the *conclusion* or that which is asserted to follow from the hypothesis.

Thus the theorem cited in § 146 may be divided into two parts as follows:

Hypothesis. Given any point P on the bisector of an angle.

Conclusion. The point P is equally distant from the sides of the angle.

151. The *formal arrangement* of a proof is exhibited in the following example:

THEOREM. *If two circles in a plane cut each other in two points, the straight line drawn through their centers bisects their common chord.*

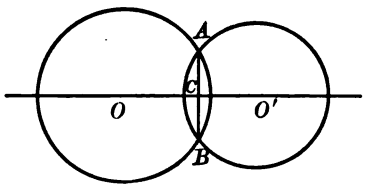


FIG. 58.

Given two circles with centers O and O', cutting each other in A and B, the straight line OO', and the common chord AB.

To prove that OO' bisects AB.

	(*Assertions*)	(*Reasons*)
Proof.	1. Imagine the figure to turn over on OO' as an axis.	
	2. Then each of the circles will again coincide with its original position.	For in a plane one circle, and only one, can be described with a given center and a given radius (§ 22).
	3. Therefore points A and B will exchange places.	For two circles can cut each other in no more than two points (§ 26, II).
	4. Therefore, the figure is symmetrical with respect to OO' as an axis, and A, B are corresponding points.	By definition (§§ 37, 38).
	5. Therefore OO' bisects AB.	For a segment joining corresponding points is bisected by the axis (§ 43).

152. It should be noted that a formal arrangement of a proof consists of three parts as follows:

Given: Here is stated, with reference to the figure which accompanies the proof, the hypothesis, or the conditions given in the theorem.

To prove: Here is stated, with reference to the figure, the conclusions asserted in the theorem.

Proof. Here is set forth, in concise steps, the argument used to prove the conclusions just stated.

It is important to observe that every assertion in the argument must be based on the hypothesis, a definition, a fundamental proposition, a construction, or a theorem previously proved. In general, the authorities supporting any such assertion should be explicitly stated in connection with the assertion. Thus, in the preceding paragraph the authorities supporting the assertions in steps 2, 3, 4, 5 are written on the right of and opposite to the respective assertions. There are, however, a considerable number of fundamental propositions either so elementary in nature that explicit reference to them each time may generally be omitted in a first course, as "if A, B, C are three points on a straight line, one and only one of the points is between the other two"; or whose applications are so apparent that explicit reference to them each time would be burdensome, as "A straight line can be drawn between any two given points," "a circle can be described from any given point as a center and with any given line segment as a radius," the fundamental propositions on motion (§§ 29–31), of which step 1 in the preceding paragraph is an example, etc.

153. No statement in a proof is warranted simply because it appears to be true from the figure, no matter how carefully drawn. The fallacies into which one may be led by trusting to appearances may be appreciated from the following examples:

In (1) and (2), Fig. 59, which appears the longer, AB or CD?

In (3) and (4), do AB and CD appear to be in the same straight line?

Test your judgments with a ruler.

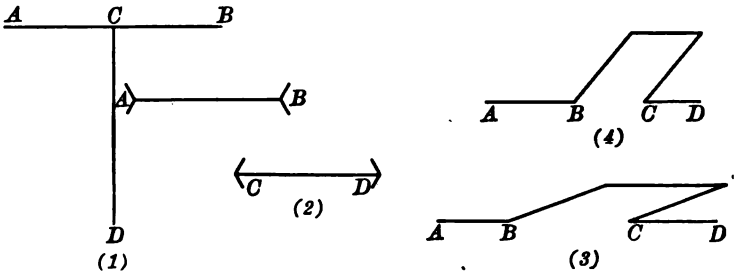


FIG. 59.

154. The following are fundamental propositions of a general nature frequently used in geometry. In order to distinguish them from fundamental propositions of a purely geometric nature they are called *axioms*. It will be convenient to refer to them by number as Axiom 1. Axiom 2, etc.

155. AXIOMS.

1. Magnitudes which are equal to the same magnitude, or to equal magnitudes, are equal to each other.

2. If equals are added to equals, the sums are equal.

3. If equals are subtracted from equals, the remainders are equal.

4. If equals are multiplied by equals, the products are equal.

5. If equals are divided by equals, the quotients are equal.

6. A magnitude may be substituted for an equal magnitude in any process.

7. If equals are added to unequals, the sums are unequal in the same sense.

8. If equals are subtracted from unequals, the remainders are unequal in the same sense.

9. If three magnitudes are so related that the first is greater than the second, while the second is greater than or equal to the third, then the first is greater than the third.

10. Any magnitude is greater than any of its parts.

156. Use may be made of the definitions and propositions of Chapter I to make clear the considerations of §§ 143–152.

PARALLEL LINES

157. Definitions. Two unlimited straight lines in the same plane which do not meet are said to be *parallel* (∥) (§ 76).

Line segments in parallel lines are also said to be *parallel.*

158. FUNDAMENTAL PROPOSITION. *Through a given point not on a given straight line, one straight line, and only one, can be drawn parallel to that line* (§ 78).

159. Preliminary theorems from Chapter I.

1. *Two straight lines which are parallel to the same straight line are parallel to each other* (§ 81).

2. *Two straight lines in the same plane which are perpendicular to the same straight line are parallel* (§ 77).

3. *A straight line which is perpendicular to one of two parallel lines is perpendicular to the other also* (§ 82).

4. *Two straight lines which are respectively perpendicular to two intersecting lines must meet* (§ 83).

5. *Parallel lines are everywhere equally distant* (§ 84).

160. A transversal. A straight line which cuts two or more lines of a figure is called a *transversal* of the figure.

161. Angles formed by a transversal of two straight lines. If a transversal EF cuts two straight lines AB and CD, the angles formed are named as follows: α, β, γ', δ' are called *exterior* angles; α', β', γ, δ are called *interior* angles; α and γ', β and δ' are called *alternate-exterior* angles; α' and γ, β' and δ are called *alternate-interior* angles; α and α', β and β', γ and γ', δ and δ' are called *corresponding* angles.

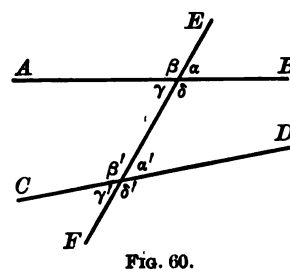


FIG. 60.

162. THEOREM. *When two straight lines in a plane are cut by a transversal, if a pair of alternate-interior angles are equal, the two straight lines are parallel.*

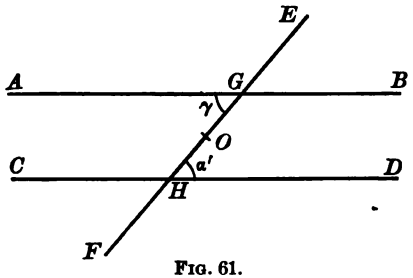


FIG. 61.

Given two straight lines AB and CD cut by the transversal EF in G and H, respectively, forming equal alternate-interior angles α' and γ.

To prove that $AB \parallel CD$.

Proof. 1. Imagine the figure to rotate in the plane about O, the mid-point of the segment GH, until G and H exchange places.

2. Then GA and HD will exchange places, (for $\alpha' = \gamma$, by hyp.).

3. $\therefore$ HC and GB will also exchange places. § 3

4. $\therefore$ the figure is symmetric with respect to O as a center. Def. § 132

5. $\therefore$ if AB and CD meet on one side of EF, they must also meet on the other side.

6. $\therefore$ AB and CD cannot meet. § 6

7. $\therefore$ $AB \parallel CD$. Def. § 157

In steps 3, 4, 6, 7 the reason for each conclusion is merely indicated at the right. The student should state the reasons in full, if possible without looking up the reference.

163. THEOREM. *When two straight lines in a plane are cut by a transversal, if a pair of corresponding angles are equal, the two straight lines are parallel.*

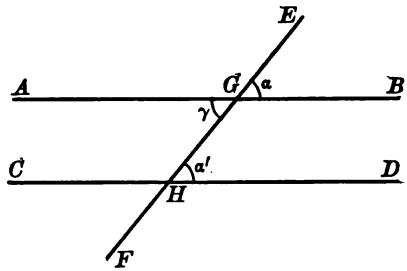


FIG. 62.

Given two straight lines AB and CD cut by the transversal EF, forming equal corresponding angles α and α'.

To prove that $AB \parallel CD$.

Proof. 1. $\gamma = \alpha$.

(For α and γ are vertical angles, § 130.)

2. But $\alpha' = \alpha$.

(By hypothesis.)

3. $\therefore \gamma = \alpha'$.

(By axiom I.)

4. $\therefore AB \parallel CD$. § 162

Exercise. When two straight lines in a plane are cut by a transversal, if a pair of alternate-exterior angles are equal, the two straight lines are parallel. See form for written proofs, § 151.

Given two straight lines, etc. . . . forming equal alternate-exterior angles α and γ'.

To prove that, etc.

Proof. 1. $\alpha' = \gamma'$. (Why ?)

2. But $\alpha = \gamma'$. (Why ?)

3. $\therefore \alpha = \alpha'$. (Why ?)

4. $\therefore AB \parallel CD$.

164. THEOREM. *When two straight lines in a plane are cut by a transversal, if the interior angles on one side of the transversal are supplementary, the lines are parallel.*

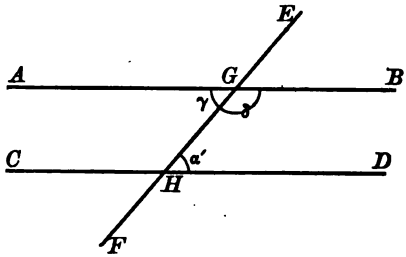


FIG. 63.

Given two straight lines AB and CD cut by the transversal EF, forming on one side of the transversal supplementary interior angles α' and δ.

To prove that $AB \parallel CD$.

Proof.	1.	γ and δ are supplementary.	§ 121
	2.	But α' and δ are supplementary.	Why?
	3.	$\therefore \alpha' = \gamma$.	§ 127
	4.	$\therefore AB \parallel CD$.	Why?

165. COROLLARY. *When two straight lines in a plane are cut by a transversal, if the sum of the interior angles on one side of the transversal is less than two right angles, the two straight lines must meet* (§ 158).

Exercise. When two straight lines in a plane are cut by a transversal, if a pair of exterior angles on the same side of the transversal are supplementary, the two straight lines are parallel.

Let the student prepare his own form and fill in the proof, according to the plan:

Given:
To prove:
Proof: (1)
(2), etc.

166. THEOREM. *When two parallel lines are cut by a transversal, any two alternate-interior angles are equal.*

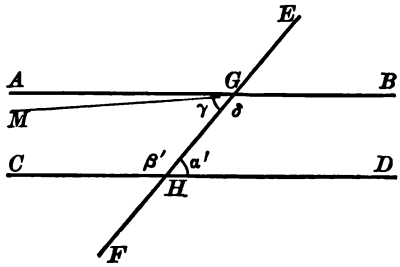


FIG. 64.

Given two parallel lines AB and CD cut by the transversal EF in G and H, respectively, forming the alternate-interior angles α', γ and β', δ as shown.

To prove that $\alpha' = \gamma$, and $\beta' = \delta$.

Proof. 1. Suppose $\gamma > \alpha'$, and that MG is drawn making $\angle MGH = \alpha'$.

2. Then MG would be parallel to CD. § 162

3. But $AB \parallel CD$. Hyp.

4. $\therefore$ the supposition that $\gamma > \alpha'$ leads to a contradiction. § 158
Likewise the supposition that $\gamma < \alpha'$ leads to a contradiction.

5. $\therefore \alpha' = \gamma$.
Similarly it can be proved that $\beta' = \delta$.

The student should carry out in detail the corresponding proof that $\beta' = \delta$.

167. COROLLARY. *The figure formed by two parallel lines and a transversal is symmetric with respect to the mid-point of the segment of the transversal intercepted by the two parallels.*

168. THEOREM. *When two parallel lines are cut by a transversal, any two corresponding angles are equal.*

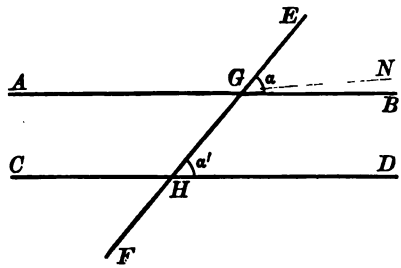


FIG. 65.

(The proof, which is similar to that in § 166, is left as an exercise.)

169. THEOREM. *If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.*

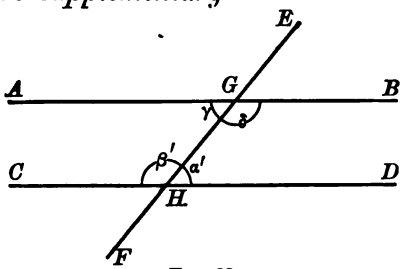


FIG. 66.

Given two parallel lines AB and CD cut by the transversal EF in G and H, forming the interior angles α', β', γ, δ, as shown.

To prove that α' and δ, and also β' and γ, are supplementary.

Proof. 1. δ and γ are supplementary. Why?

2. But $\alpha' = \gamma$. Why?

3. $\therefore$ α' and δ are supplementary, Ax. 6

In like manner it can be proved that β' and γ are supplementary.

170. THEOREM. *If the sides of an angle are parallel respectively to sides of another angle, the two angles are either equal or supplementary.*

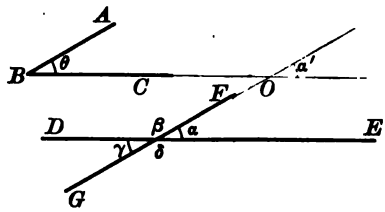


FIG. 67.

Given the angle $ABC (= \theta)$ and the lines DE and GF, with $DE \parallel BC$ and $GF \parallel BA$, forming the angles as shown.

To prove that α and γ are equal to θ, and that β and δ are supplementary to θ.

Proof. 1. Let BC and GF meet in O, forming the angle α' as indicated.

2. Then	$\alpha = \alpha'$, and $\theta = \alpha'$.	Why?
3.	$\therefore \alpha = \theta$.	Why?
4. Also	$\gamma = \alpha$.	Why?
5.	$\therefore \gamma = \theta$.	
6. Moreover β and α are supplementary.		Why?
7. $\therefore \beta$ and θ are supplementary.		Why?
8. Also, δ and θ are supplementary.		Why?

171. COROLLARY. *If the sides of an angle are perpendicular respectively to the sides of another angle, the angles are either equal or supplementary.*

(The proof is left as an exercise.)

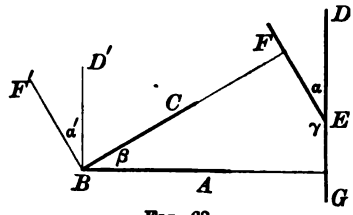


FIG. 68.

172. PROBLEM. *Through a given point not on a given straight line, to draw a straight line parallel to that line.* (Second Method.*)

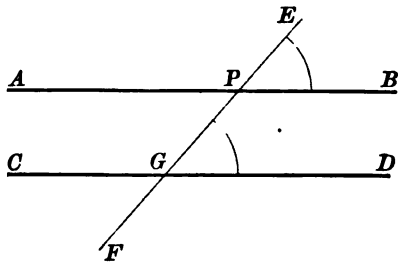


FIG. 69.

Given the straight line CD and the point P not on CD.

Required to draw a straight line through P parallel to CD.

Construction. Draw a straight line PF cutting CD in G; and through P draw AB making $\angle EPB = \angle PGD$. § 117

Then AB is the required line.

Proof. 1.	$\angle EPB = \angle PGD$.	Construction
2.	$\therefore AB \parallel CD$	Why?

173. Note. The above construction may be made much more expeditiously by the use of a draftsman's triangle and a straightedge, as follows:

Adjust the straightedge and triangle as shown in Fig. 70, so that MN, one edge of the triangle, coincides with the given line CD. Then, keeping the straightedge fixed, slide the triangle along the latter until the edge MN passes through P.

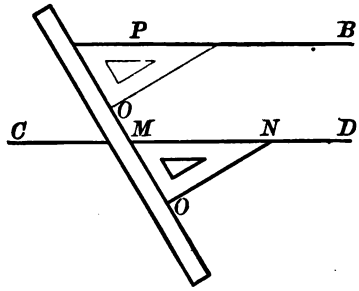


FIG. 70.

Trace a line along this edge. This is the required line. Why?

* See § 79 for other method.

CONVERSE PROPOSITIONS

174. The student should now note that the theorems proved in the preceding sections relating to the figure formed by two lines and a transversal go in pairs that bear a certain mutual relation. For example, in § 162 we proved that

If two alternate-interior angles are equal, then the lines are parallel; whereas in § 166 it was shown that

If the two lines are parallel, then the alternate-interior angles are equal.

It is clear that the hypothesis of the first theorem is the conclusion of the second, while the conclusion of the first theorem is the hypothesis of the second. Let the student verify for himself that the theorems in §§ 163 and 168 are similarly related; as are also the theorems in §§ 164 and 169.

If two propositions are so related that the hypothesis of the one is the conclusion of the other, and *vice versa*, the propositions are said to be *converse propositions* and either is said to be the *converse* of the other.

Thus the theorems quoted above from §§ 162 and 166 are converse propositions, and each is the converse of the other.

The converse of a given proposition may always be formed, but it MAY NOT be a *true* proposition. For example, we have seen (§ 170) that *if the sides of an angle are respectively parallel to the sides of another angle, the two angles are equal or supplementary.* The converse statement is that if two angles are equal or supplementary, the sides of one must be parallel respectively to the sides of the other. This statement is obviously not true. *The converse of a theorem must therefore always be proved if it is to be asserted as a new theorem.*

175. * If the hypothesis of a theorem consists of more than one part, it may have more than one converse. Consider, for example, the theorems (§ 113): *In equal circles equal arcs are subtended by equal chords.* The hypothesis and conclusion may here be exhibited as follows:

* This section may be omitted at the discretion of the teacher.

HYPOTHESIS. (*a*) If two circles are equal,
and
(*b*) If two arcs are equal,
then

CONCLUSION. (*c*), the chords subtending the arcs are equal.

Here we may form a new proposition by interchanging either (*b*) and (*c*), or (*a*) and (*c*). In the first case we obtain the proposition:

If in two equal circles two chords are equal, the subtended arcs are equal.

In the second case we obtain the proposition:

If in two circles two equal chords subtend equal arcs, the circles are equal.

Each of the last two propositions is *a converse* of the original. (They both happen to be true.)

To summarize the discussion of this section:

If a given proposition has a hypothesis consisting of two or more parts, any proposition obtained from the given one by interchanging the conclusion and a part of the hypothesis is a converse of the given proposition.* Such a converse may or may not be true; *it must never be asserted without proof.*

176. EXERCISES

1. State and prove the converse of the exercise on p. 54.

2. Show that Proposition 3 of § 159 is a consequence of any one of the following: §§ 166, 168, 169.

3. One of the angles formed by a transversal cutting two parallel lines is 40°. Find all the other angles.

4. *State* the converse of each of the propositions cited below: § 26, I; § 34; § 48; § 51; § 54; § 58; § 67; § 74.

Which of these converses do you think are true? Which do you know to be false? Is any one of the propositions above its own converse?

* We have assumed that the conclusion has only one part. Any proposition in which the conclusion has more than one part consists, strictly speaking, of several separate propositions, each having a single conclusion.

CHAPTER III

TRIANGLES

177. Collinear points. If three or more points lie on the same straight line, they are said to be *collinear*. If the points do not lie on the same straight line, they are said to be *non-collinear*.

178. A triangle. The figure formed by the line segments joining three non-collinear points is called a *triangle* ($\triangle$).

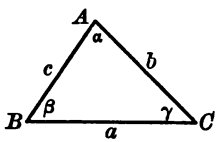


FIG. 71.

In the triangle ABC the points A, B, C are called the *vertices*, the segments AB, BC, CA are called the *sides*, and the angles ABC, BCA, CAB are called the *angles* of the triangle.

179. Notation. It is customary to represent the vertices of a triangle by capital letters, as A, B, C; to denote the corresponding angles by the symbols $\angle A$, $\angle B$, $\angle C$, or by the Greek letters α, β, γ, respectively, and to designate the sides by the corresponding small letters a, b, c, or by the notation "side BC," "side AC," "side AB," respectively. When a, b, c are used to denote the sides, it is customary to denote by a the side opposite $\angle A$, by b the side opposite $\angle B$, and by c the side opposite $\angle C$ (Fig. 71).

Two angles β and γ are said to be *adjacent* to the common side a; the angles β and γ are also said to *include* the common side a.

Two sides b and c are said to *include* the common adjacent angle α, and the other side a is said to be *opposite* α.

180. THEOREM. *The sum of the angles of a triangle is equal to two right angles* ($2R$).

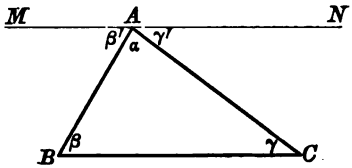


FIG. 72.

Given the triangle ABC, with the angles α, β, γ.

To prove that $\alpha + \beta + \gamma = 2\,R$.

Proof. 1. Draw MN through A parallel to BC, forming angles β' and γ' as shown.

2. Then, $\beta' + \alpha + \gamma' = 2\,R$ § 123, I

3. But $\beta = \beta'$ and $\gamma = \gamma'$ (Why ?)

(Let the student complete the proof.)

181. COROLLARY 1. *The sum of two angles of a triangle is less than two right angles.*

182. COROLLARY 2. *A triangle cannot have more than one right angle or one obtuse angle.*

183. COROLLARY 3. *If the sum of two angles of a triangle is taken from two right angles, the remainder is equal to the third angle.*

184. COROLLARY 4. *If two triangles have two angles of the one equal to two angles of the other, the third angles are equal.*

185. An exterior angle of a triangle. An angle formed by one side of a triangle and an adjacent side produced through the common vertex is called an *exterior angle of the triangle.* Thus DCA (Fig. 73) is an exterior angle of the triangle ABC. The angles α and β are called *opposite interior angles,* and, of course, γ is *adjacent* to the exterior angle DCA.

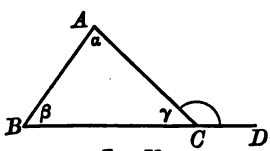


FIG. 73.

186. THEOREM. *An exterior angle of a triangle is equal to the sum of the opposite interior angles.*

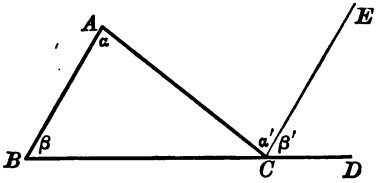
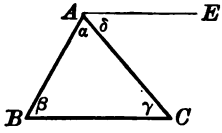


FIG. 74.

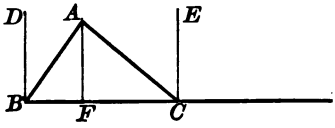
(The proof is left to the student.)

187. COROLLARY. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

188. **EXERCISES**



1. Prove the theorem of § 180, using the adjacent figure. See § 169. AE is drawn parallel to BC. Prove it also by using the figure of § 186.



2. Prove the theorem of § 180, using the adjacent figure. BD, CE, and AF are drawn perpendicular to BC.

3. If the three angles of a triangle are equal, how many degrees in each?

4. If one angle of a triangle is 35° 30′, and the other angles are equal, find the value of each.

5. Two angles of a triangle are, respectively, three times and four times the remaining angle. Find the value of each angle.

6. If an exterior angle of a triangle is 45° and the opposite interior angles are equal, find the value of each angle of the triangle.

7. If two angles of a triangle are 40° and 60°, find the angles formed by their bisectors.

8. One angle of a triangle is 36°. Find the other angles of the triangle if one of them is twice the other.

CONGRUENT TRIANGLES. GENERAL THEOREMS

189. THEOREM. *Two triangles are congruent if two angles and the included side of the one are equal, respectively, to two angles and the included side of the other.*

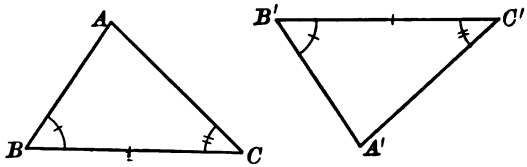


FIG. 75.

Given the $\triangle$s ABC and $A'B'C'$, with $\angle B = \angle B'$, side BC = side $B'C'$, and $\angle C = \angle C'$.

To prove that $\triangle ABC \equiv \triangle A'B'C'$.

Proof. 1. Imagine $\triangle A'B'C'$ to slide along the line $B'B$ until B' coincides with B; and then to rotate about B until side $B'C'$ coincides with its equal side BC. If then the triangles are not on the same side of BC, turn $\triangle A'B'C'$ over on BC as an axis.

2. Then the line $B'A'$ will coincide with the line BA, (for $\angle B' = \angle B$, by hyp.).

3. And the line $C'A'$ will coincide with the line CA, (for $\angle C = \angle C'$, by hyp.).

4. $\therefore A'$ will coincide with A. § 6

5. $\therefore \triangle ABC \equiv \triangle A'B'C'$. Def. § 32

190. COROLLARY. *Two triangles are congruent if two angles and any side of the one are equal, respectively, to two angles and the corresponding side of the other.* § 184

NOTE. By "corresponding" side is here meant the side opposite to that angle of the second triangle which is equal to the angle opposite the given side in the first triangle.

191. THEOREM. *Two triangles are congruent if two sides and the included angle of the one are equal, respectively, to two sides and the included angle of the other.*

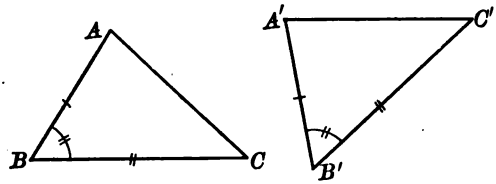


FIG. 76.

Given the $\triangle$s ABC and $A'B'C'$ with side $AB =$ side $A'B'$, side $BC =$ side $B'C'$, and $\angle B = \angle B'$.

To prove that $\triangle ABC \equiv \triangle A'B'C'$.

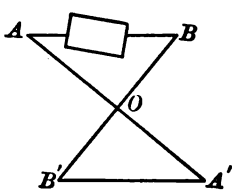
Proof. 1. Imagine $\triangle A'B'C'$ to slide along the line $B'B$ until B' coincides with B; and then to rotate about B until side $B'C'$ coincides with its equal side BC. If then the triangles are not on the same side of BC, turn $\triangle A'B'C'$ over on BC as an axis.

2. Then $B'A'$ will coincide with BA,
(for $\angle B = \angle B'$, by hyp.).

3. And A' will coincide with A.
(for side $B'A' =$ side BA, by hyp.).

4. $\therefore A'C'$ will coincide with AC. § 3

5. $\therefore \triangle ABC \equiv \triangle A'B'C'$. Def. § 32



Exercises. 1. Show how to find the distance between two points A and B when the line of sight between A and B is obstructed, each being visible from a third point O (§ 191).

2. Prove that if the sides of two triangles are respectively parallel the angles of one triangle will be equal respectively to the angles of the other. Are the triangles necessarily congruent?

192. THEOREM. *Two triangles are congruent if the three sides of the one are equal, respectively, to the three sides of the other.*

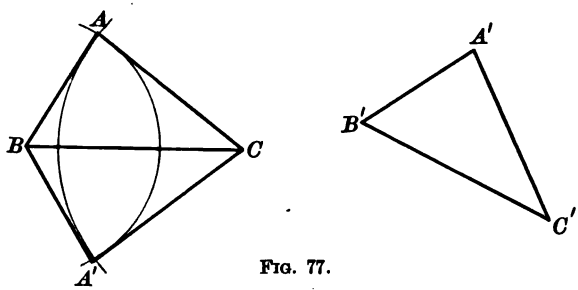


FIG. 77.

Given the $\triangle$s ABC and $A'B'C'$ with sides AB, BC, CA equal, respectively, to sides $A'B'$, $B'C'$, $C'A'$.

To prove that $\triangle ABC \equiv \triangle A'B'C'$.

Proof. 1. Imagine $\triangle A'B'C'$ to slide along the line $B'B$ until B' coincides with B; and then to rotate about B until side $B'C'$ coincides with its equal side BC. If then the triangles are on the same side of BC, turn $\triangle A'B'C'$ over on BC as an axis.

2. Since side BA = side BA', and side CA = side CA', Why?
circles may be described from centers B and C which will pass through A and A'. § 25

3. $\therefore$ if the figure is turned over on BC as an axis, the triangles will exchange places. Why?

4. $\therefore \triangle ABC \equiv \triangle A'B'C'$. Why?

193. Homologous parts of congruent figures. In congruent figures the line segments, angles, etc., which coincide when either figure is brought into coincidence with the other are called *homologous parts.*

In congruent triangles homologous sides are opposite equal angles, and conversely.

194. Example. An ancient geometer measured the distance from a point A on land to a ship S in the following manner: Choosing a second point B visible from A and S, he located two stations P and Q by sighting along AS and BS, respectively. Then he located the lines AS' and BS', making $\alpha' = \alpha$, and $\beta' = \beta$. Finally he measured the distance AS'. Justify the method.

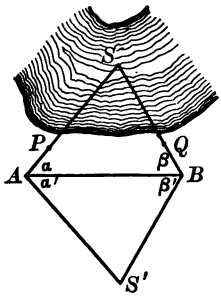
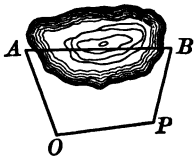


FIG. 78.

Triangles ABS' and ABS are evidently congruent, since two angles and the included side of the one are equal respectively to two angles and the included side of the other (§ 189). Therefore, we conclude immediately that $AS' = AS$, since these sides are opposite the equal angles β and β' in the two triangles.

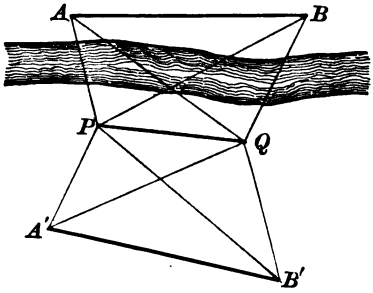


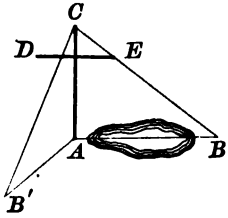
Exercises. 1. Show how to find the distance between two points A and B when the line of sight between A and B is obstructed and no third point can be found which is visible from both A and B.

Find points O and P which are visible, each from the other, and from A and B, respectively.

2. Show how to find the distance between two inaccessible points, as the distance between two points A and B on the opposite side of a river from the observer.

Choose two points P, Q visible from A and B and lay out a symmetric figure having PQ as an axis, or invent an original method.





3. In early days a device consisting of a vertical staff AC to which was attached a cross bar DE, that could be moved up and down on the staff, was used in measuring distances. Discover how the distance from A to an inaccessible point B could be measured with such an instrument.

THE ISOSCELES TRIANGLE

195. Definitions. If no two sides of a triangle are equal, it is called a *scalene triangle;* if two sides are equal, it is called an *isosceles triangle;* and if the three sides are equal, it is called an *equilateral triangle.*

The angle included by the equal sides of an isosceles triangle is called the *vertical angle of the isosceles triangle,* and the side opposite this angle is called the *base.*

196. THEOREM. *In an isosceles triangle the angles opposite the equal sides are equal.*

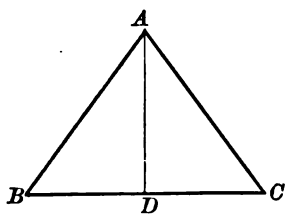


FIG. 79.

Given the $\triangle ABC$ with $AB = AC$.

To prove that $\angle C = \angle B$.

Proof. 1. Draw AD bisecting $\angle BAC$ and cutting BC in D.

2. Then in $\triangle\triangle$ ABD and ACD

AD is common,

$AB = AC$, Why?

and $\angle BAD = \angle CAD$. By construction

3. $\therefore \triangle ABD \equiv \triangle ACD$. § 191

4. $\therefore \angle B = \angle C$, § 193

(being homologous parts of congruent triangles).

197. COROLLARY 1. *If the sides of a triangle are equal, the angles are also equal.*

198. COROLLARY 2. *An isosceles triangle is symmetric with respect to the bisector of the angle included by the equal sides.*

199. THEOREM. *If two angles of a triangle are equal, the sides opposite those angles are equal.*

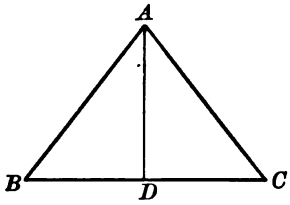


FIG. 80.

Given the $\triangle ABC$ with $\angle B = \angle C$.

To prove that $AC = AB$.

Proof. 1. Draw AD bisecting $\angle BAC$ and cutting BC in D.

2. Then in $\triangle$s ABD and ACD

AD is common,

$\angle B = \angle C$, Why?

and $\angle BAD = \angle CAD$. Why?

$\therefore \angle ADB = \angle ADC$. § 184

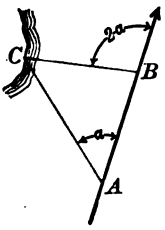
3. $\therefore \triangle ABD \equiv \triangle ACD$. § 190

4. $\therefore AB = AC$, § 193

(being homologous parts).

200. COROLLARY. *If the angles of a triangle are equal, the triangle is equilateral.*

Exercise. The following is a mariner's rule for estimating the distance of an object from a ship: Let the ship be moving in the direction AB. Observe the moment the direction of the object C makes a given angle with the course AB, and again when it makes an angle twice as large. The distance BC will then be equal to AB, the distance through which the ship has moved between the two observations. Prove it.



201. Definitions. The three perpendiculars from the vertices of a triangle to the opposite sides (produced if necessary) are called the *altitudes* (Fig. 81); the three line segments joining the vertices to the middle points of the opposite sides are called the *medians* (Fig. 82); and the three line segments bisecting the angles and terminating in the opposite sides are called the *bisectors of the angles* of the triangle (Fig. 83).

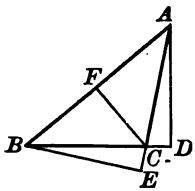


FIG. 81.

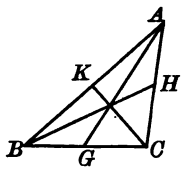


FIG. 82.

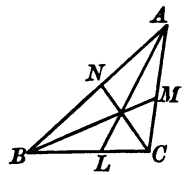
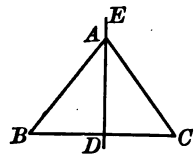


FIG. 83.

NOTE. It should be noted that the altitudes, medians, and bisectors of the angles of a triangle as here defined are *line segments;* the lengths of these segments are also referred to by the words altitude, median, and bisector. With regard to the latter term it should be noted, in addition, that the "bisector of an angle" is also used to denote a straight line not limited in length (see § 72); whereas here it is used to denote a segment associated with a triangle. The context will always indicate which meaning is to be attached to the term.

202. **EXERCISES**

1. If the perpendicular bisector of one side of a triangle passes through the vertex of the opposite angle, the triangle is isosceles.



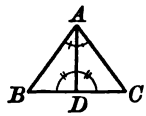
Given the triangle ABC with DE, the perpendicular bisector of BC, passing through the opposite vertex A.

To prove that $\triangle ABC$ is isosceles.

Proof.		(*Reasons*).
1.	In the $\triangle$s ABD and ACD	
	AD is common,	
	$DB = DC$,	(?)
	and $\angle ADB = \angle ADC$.	(?)
2.	$\therefore \triangle ABD \equiv \triangle ACD$.	(?)
3.	$\therefore AB = AC$.	(?)
4.	$\triangle ABC$ is isosceles.	(?)

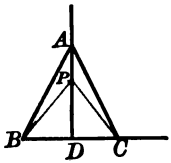
2. Prove Ex. 1 by symmetry. Imagine the figure to turn over on DE as an axis.

3. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.



1. In the $\triangle$s ABD and ACD, AD is common, $\angle BAD = \angle CAD$ (?), and $\angle BDA = \angle CDA$ (?).
2. $\therefore \triangle ABD \equiv \triangle ADC$ (?), etc.

4. Prove Ex. 3 by symmetry.



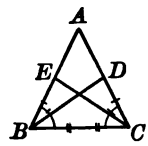
5. Every point on the bisector of the angle included by the equal sides of an isosceles triangle is equidistant from the extremities of the base.

HINT: $\triangle ABP \equiv \triangle ACP$. $\therefore PB = PC$.

6. Prove Ex. 5 by symmetry.

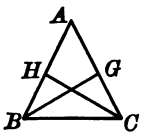
7. The bisector of the angle included by the equal sides of an isosceles triangle bisects the base and is perpendicular to the base.

8. The bisector, the median, and the altitude drawn to the base of an isosceles triangle are congruent. What can be said about the altitudes, medians, and bisectors of an equilateral triangle?



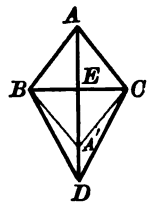
9. The medians drawn to the equal sides of an isosceles triangle are equal and make equal angles with the base.

HINT: $\triangle BCE \equiv \triangle CBD$.



10. The bisectors of the angles adjacent to the base of an isosceles triangle are equal.

HINT: $\triangle BCH \equiv \triangle CBG$.



11. If two isosceles triangles have the same base, the straight line passing through the vertices opposite the common base is perpendicular to the base and bisects it.

HINT: $\triangle ABD \equiv \triangle ACD$. $\therefore \angle BAE = \angle CAE$.

12. Prove Ex. 11 by symmetry.

13. Why is a wooden gate sufficiently braced, theoretically, by a timber running from the corner nearest the foot of the post to the upper and outer corner of the gate? Why is it that in practice such a brace is sometimes found insufficient?

RIGHT TRIANGLES

203. Definitions. A triangle is called a *right triangle* (Rt. △) if it has a right angle, an *obtuse triangle* if it has an obtuse angle, and an *acute triangle* if all its angles are acute. Obtuse and acute triangles are classed under the general term *oblique triangle.*

The side of a right triangle opposite the right angle is called the *hypotenuse,* and the other two sides are called the *legs* of the right triangle.

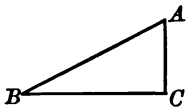


FIG. 84.

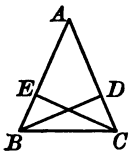
The following are immediate consequences of §§ 190, 191:

204. THEOREM. *Two right triangles are congruent if the legs of the one are equal, respectively, to the legs of the other.* § 191

205. THEOREM. *Two right triangles are congruent if the hypotenuse and an acute angle of the one are equal, respectively, to the hypotenuse and an acute angle of the other.* § 190

206. THEOREM. *Two right triangles are congruent if a leg and the adjacent (or the opposite) acute angle of the one are equal, respectively, to a leg and the adjacent (the opposite) acute angle of the other.* § 190

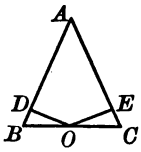
207. EXERCISES



1. The altitudes drawn to the equal sides of an isosceles triangle are equal.

Rt. △ $BEC \equiv$ Rt. △ BDC.

2. State and prove the converse of Ex. 1.



3. The perpendiculars drawn from the middle point of the base to the equal sides of an isosceles triangle are equal.

208 THEOREM. *Two right triangles are congruent if the hypotenuse and a leg of the one are equal, respectively, to the hypotenuse and a leg of the other.*

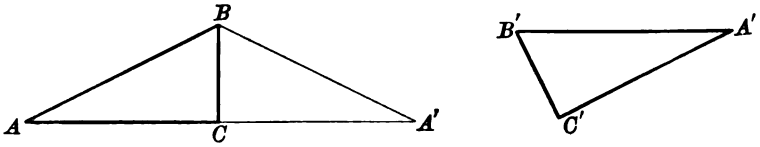


FIG 85.

Given the Rt. $\triangle$ ABC and $A'B'C'$ with Rt. $\angle$ C and C', and with $AB = A'B'$ and $BC = B'C'$.

To prove that Rt. $\triangle$ $ABC \equiv$ Rt. $\triangle$ $A'B'C'$.

Proof. 1. Imagine $\triangle$ $A'B'C'$ to slide along the line $C'C$ until C' and C coincide; and then to rotate about C until $C'B'$ coincides with its equal CB. If then the triangles are on the same side of BC, turn $\triangle$ $A'B'C'$ over on BC as an axis until it falls again in the plane of $\triangle$ ABC.

2. Then since $\angle$ C and C' are Rt. $\angle$ — Why?
AC and CA' form a straight line. — Why?

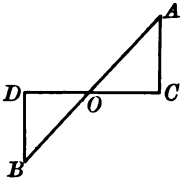
3. Also in $\triangle$ $AA'B$, $BA = BA'$. — Why?
$\therefore \angle A = \angle A'$. — Why?

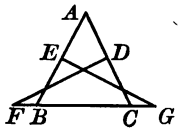
4. $\therefore$ Rt. $\triangle$ $ABC \equiv$ Rt. $\triangle$ $A'BC$. — § 206.
That is, Rt. $\triangle$ $ABC \equiv$ Rt. $\triangle$ $A'B'C'$.

209. EXERCISES

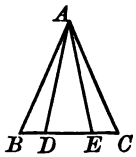
1. State and prove the converse of Ex. 3, § 207.

2. Two straight lines cut each other at a point O, and from any two points on one of the lines equally distant from O perpendiculars are drawn to the other line. Prove that these perpendiculars are equal.

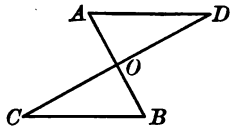




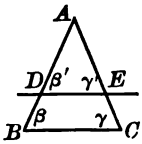
3. Perpendiculars drawn to the equal sides of an isosceles triangle at the middle points and terminating in the base or the base produced are equal.



4. If equal segments are laid off on the base of an isosceles triangle from the extremities, lines joining the ends of these segments to the opposite vertex are equal.



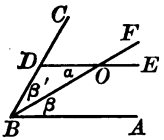
5. If two line segments bisect each other, the line segment joining any two unconnected end points of the given segments is equal and parallel to the segment joining the other two end points.



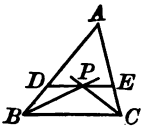
6. A straight line drawn parallel to the base of an isosceles triangle, cutting the other two sides, forms equal angles with those sides.

$\beta' = \beta$ and $\gamma' = \gamma$. But $\beta = \gamma$. $\therefore \beta' = \gamma'$.

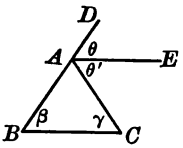
7. If a straight line cuts the equal sides of an isosceles triangle and is parallel to the base, the segments of the equal sides between the parallel and the base are equal.



8. If through any point on the bisector of an angle a straight line is drawn parallel to either of the sides, the triangle thus formed is isosceles.

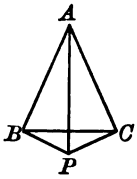


9. If the bisectors of two angles of a triangle are drawn, and through their point of intersection a line is drawn parallel to the side included by these angles, the segment of this parallel between the other two sides is equal to the sum of the segments of these sides between the parallel and the first side.



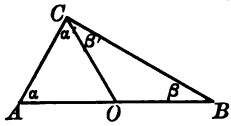
10. The bisector of an exterior angle of an isosceles triangle formed by one of the equal sides and the other equal side produced, is parallel to the base. $\theta + \theta' = \beta + \gamma$ (§ 186). But $\theta = \theta'$, and $\beta = \gamma$. $\therefore 2\theta = 2\beta$, or $\theta = \beta$. $\therefore AE \parallel BC$.

11. State and prove the converse of Ex. 10.



12. Straight lines drawn perpendicular to the equal sides of an isosceles triangle at the extremities of the base intersect on the bisector of the opposite angle.

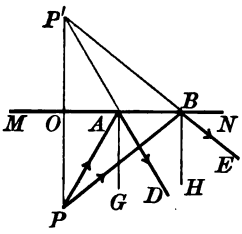
13. If the median drawn to one side of a triangle is equal to half that side, the angle opposite that side is a right angle.



$\alpha + (\alpha' + \beta') + \beta = 2R$. But $\alpha = \alpha'$, and $\beta = \beta'$.
$\therefore 2\alpha' + 2\beta' = 2R$, or $\alpha' + \beta' = R$.

14. The angle formed by the chords joining any point on a circle to the extremities of a diameter is a right angle. (Use the result of Ex. 13.)

15. It is a fact of everyday experience that the image of an object as seen in a plane mirror appears to be as far back of the mirror as the object is in front of it. To prove this suppose a ray of light having its origin at P strikes a plane mirror MN at A. Then from a well-known principle in physics the ray of light will be reflected along the line AD, such that PA and AD make equal angles with a perpendicular to MN at A. That is, "The angle of reflexion is equal to the angle of incidence." Let BE be the line of reflection of another such ray having its origin at P. Then to an observer the reflected rays AD and BE will appear to have their origin at P', the intersection of DA and EB produced. Prove that $PP' \perp MN$, and that the segment PP' is bisected by MN.



16. To cut two converging timbers by a line AB which shall make equal angles with them, a carpenter proceeds as follows: Place two squares against the timbers as shown in the adjacent figure, so that $AO = BO$. Show that AB is the required line.

INEQUALITIES

210. THEOREM. *If two sides of a triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.*

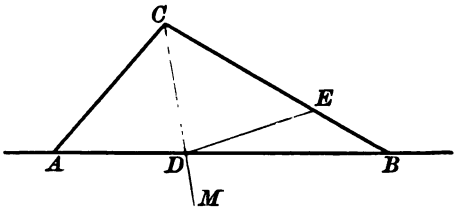


FIG. 86.

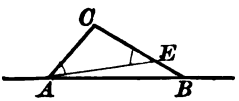
Given the $\triangle ABC$ with $CB > CA$.

To prove that $\angle A > \angle B$.

Proof. 1. Draw CM bisecting $\angle C$ and cutting AB in D. On CB lay off $CE = CA$, and draw DE.

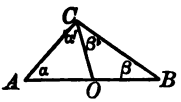
2.	Then	$\triangle CDE \equiv \triangle CDA$.	Why?
3.		$\therefore \angle CED = \angle CAD$.	Why?
4.	But	$\angle CED > \angle B$.	§ 187
5.		$\therefore \angle A > \angle B$.	Why?

211. Definition. The *opposite of any proposition* is obtained by replacing both the hypothesis and the conclusion by their opposites. Thus, the first part of § 210 is the opposite of § 196.



Exercises. 1. Prove § 210 by using the adjacent figure. Take $CE = CA$, and draw AE. $\angle CEA = \angle CAE$.

2. An angle of a triangle is acute or obtuse, according as the median through the vertex of that angle is greater than, or less than, half the opposite side. See Ex. 13, § 209.



To prove the first suppose $OC > OB$ or OA. Then $\beta > \beta'$ and $\alpha > \alpha'$.

But $\alpha + (\alpha' + \beta') + \beta = 2R$. $\therefore 2\alpha' + 2\beta' < 2R$, or $\alpha' + \beta' < R$.

212. THEOREM. *If two angles of a triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.*

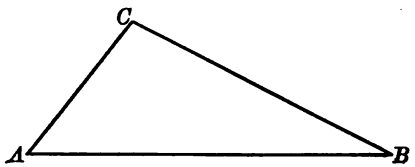


FIG. 87.

Given the $\triangle ABC$ with $\angle A > \angle B$.

To prove that $CB > CA$.

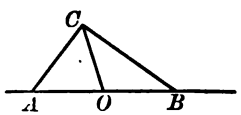
Proof. 1. If $CB = CA$, $\angle A = \angle B$. Why?

2. And if $CB < CA$, $\angle A < \angle B$. § 210

3. Both of these conclusions contradict the hypothesis. (for $\angle A > \angle B$, by hyp.).

4. $\therefore CB > CA$.

213. Proof by reductio ad absurdum. The method of proof exhibited above is a form of indirect proof called *proof by reductio ad absurdum* (reduction to an absurdity). It consists, essentially, in assuming the theorem false, and endeavoring to prove that the assumption leads to an absurdity. Thus in the above proof the assumption that $CB = CA$ leads to the absurdity that $\angle A$ is both greater than and equal to $\angle B$. Similarly the assumption that $CB < CA$ leads to the absurdity that $\angle A$ is both greater than and less than $\angle B$. Consequently both assumptions must be rejected; and the only remaining possibility is that $CB > CA$. The student should make clear to himself the nature of this kind of proof.



Exercise. A median of a triangle is greater than, equal to, or less than half the side it bisects, according as the angle opposite that side is acute, right, or obtuse. Prove by reductio ad absurdum that this is a consequence of Ex. 13, § 209, and Ex. 2, § 211.

214. THEOREM. *If two sides of a triangle are equal, respectively, to two sides of another, but the included angle of the first triangle is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*

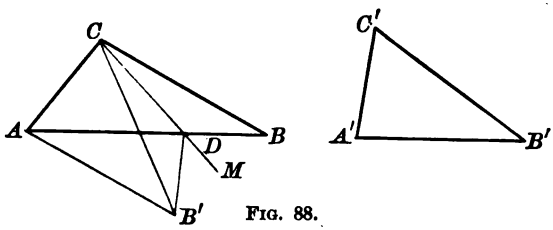


FIG. 88.

Given the $\triangle$s ABC and $A'B'C'$ with $CA = C'A'$, $CB = C'B'$, and $\angle ACB > \angle A'C'B'$.

To prove that $AB > A'B'$.

Proof. 1. Imagine $\triangle A'B'C'$ to slide along the line $C'C$ until C' coincides with C; and then to rotate about C until $C'A'$ coincides with its equal CA. If then the triangles are not on the same side of AC, turn $\triangle A'B'C'$ over on AC as an axis.

2. Then $C'B'$ will fall within $\angle ACB$. Why?

3. Draw CM bisecting $\angle BCB'$, and cutting AB in D. Draw DB'.

4. Then $\triangle BCD \equiv \triangle B'CD$. Why?

5. $\therefore DB = DB'$. Why?

6. But $AD + DB' > AB'$. Why?

7. $\therefore AD + DB > AB'$. Why?

That is, $AB > A'B'$.

NOTE. A simple form of crane consists of a movable beam hinged at the bottom to a fixed vertical mast and controlled by a rope attached to the upper end of the beam and running over a pulley at the top of the mast. When the rope is let out, the beam descends. How does this illustrate the above theorem and the following?

215. THEOREM. *If two sides of a triangle are equal, respectively, to two sides of another, but the third side of the first triangle is greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.*

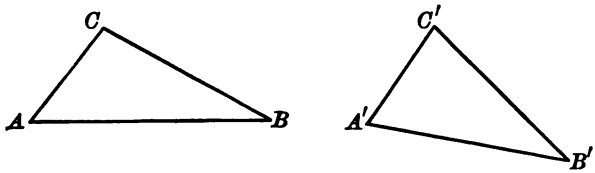


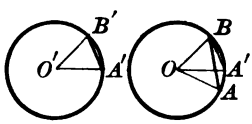
FIG. 89.

Given the $\triangle ABC$ and $A'B'C'$ with $CA = C'A'$, $CB = C'B'$, but $AB > A'B'$.

To prove that $\angle C > \angle C'$.

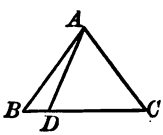
(The proof is left to the student. Prove by reductio ad absurdum.)

216. **EXERCISES**



1. In the same circle or in congruent circles, if two minor arcs are unequal, the greater arc is subtended by the greater chord.

Imagine the circle with center O' to slide along the line $O'O$ until O coincides with O. Then the circles will coincide. Then rotate the former circle about O until the arc $A'B'$ falls within arc AB. Apply § 215.



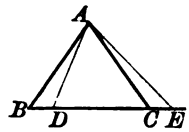
2. State and prove the converse of Ex. 1.

3. The segment joining any point on the base of an isosceles triangle to the vertex of the opposite angle is less than either of the equal sides.

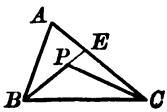
$\angle ADC > \angle B$. But $\angle B = \angle C$.

4. The segment joining any point on the produced parts of the base of an isosceles triangle to the vertex of the opposite angle is greater than either of the equal sides.

5. Not more than two equal segments can be drawn from a point to a straight line.



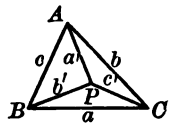
6. The difference of two sides of a triangle is less than the third side. If a, b, c are the sides, then $a + b > c$. $\therefore b > c - a$.

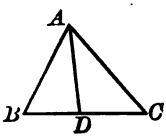


7. The sum of two sides AB, AC of a triangle ABC is greater than the sum of the segments joining a point P within the triangle to the end points of the third side BC.

Produce BP to E. Since $PE + EC > PC$, $BE + EC > BP + PC$. But $AB + AE > BE$, etc.

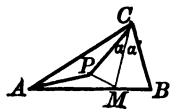
8. The sum of the segments which join a point within a triangle to the three vertices is less than the sum of the sides, but greater than half the sum.

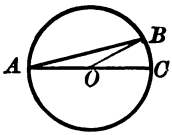




9. The bisector of the angle A of a triangle ABC cuts BC in D. Prove that $BA > BD$, and $CA > CD$.

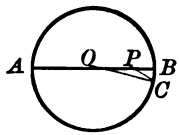
10. P is a point within a triangle ABC, such that $PC = CB$. Prove that $AB > AP$.





11. A diameter is greater than any other chord of a circle.

12. The shortest line that can be drawn to a circle from a point within it is the shorter segment of the diameter through that point.



Let P be the given point. $OP + PC > OC$. But $OB = OC$.

13. The longest line that can be drawn to a circle from a point within it is the longer segment of the diameter through the point.

14. State and prove theorems similar to Exs. 12 and 13, when the point P is outside the circle.

CONSTRUCTION PROBLEMS

217. Parts of a triangle. The sides, angles, altitudes, medians, etc., are called the *parts* of the triangle.

218. Determining parts of a figure. A figure is said to be *determined* by certain given parts when all figures constructed with the given parts are congruent. Hence:

219. *A triangle is determined,*

1. *When two sides and the included angle are given.* § 191
2. *When the three sides are given.* § 192
3. *When two angles and any side are given.** § 190

A right triangle is determined,

4. *When two sides are given.* §§ 204, 208
5. *When a side and one acute angle are given.* §§ 205, 206

Exercise. Is the following proposition correct? Two triangles are congruent if any three parts (sides, angles) of one triangle are equal respectively to the three corresponding parts of the other, provided only that at least one of the parts is a side. Why?

220. PROBLEM. *To construct a triangle, given two sides and the included angle.*

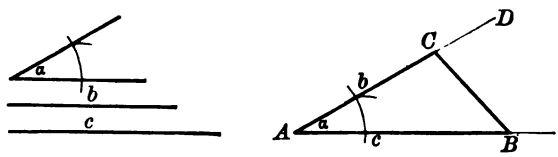


FIG. 90.

Given the sides b, c, and the included angle α.

Required to construct $\triangle ABC$.

Construction. Draw AB equal to c. At A construct $\angle BAD$ equal to α (§ 117). On AD lay off AC equal to b and draw BC.

Then $\triangle ABC$ is the triangle required.

Proof. 1. $AB = c$, $\angle BAC = \alpha$, and $AC = b$. Construction

2. $\therefore \triangle ABC$ is constructed. § 219, 1

* A side is not "given" unless, in addition to its length, it is known opposite what angle it is.

221. PROBLEM. *To construct a triangle, given the three sides.*

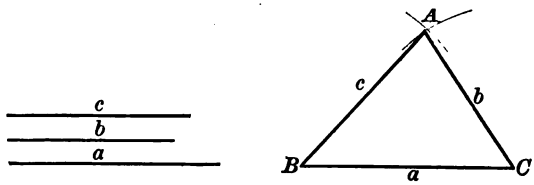


FIG. 91.

Given the sides a, b, c.

Required to construct $\triangle ABC$.

Construction. Draw BC equal to a. From B and C as centers, with radii c and b, respectively, describe arcs cutting each other in A. Draw BA and CA.

Then $\triangle ABC$ is the triangle required.

Proof. 1. $BC = a$, $AB = c$, and $CA = b$. Construction

2. $\therefore \triangle ABC$ is constructed. § 219, 2

Discussion. If the sum, or difference, of two sides is equal to the third side, or if the sum of two sides is less than the third side, no solution is possible. Why?

222. PROBLEM. *To construct a triangle, given two angles and the included side.*

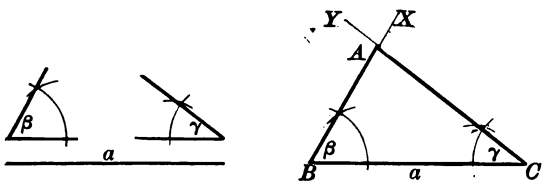


FIG. 92.

Given the angles β and γ and the included side a.

Required to construct $\triangle ABC$.

Construction and Proof. These are left to the student.

Discussion. If $\beta + \gamma = 2R$, or $> 2R$, no solution is possible. Why?

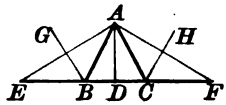
Exercises. 1. To construct an equilateral triangle, given a side.

2. To construct an angle of 60°; 30°; 120°.

3. To construct an equilateral triangle, given an altitude.

4. To construct an isosceles triangle, given:
(*a*) the base and the altitude on the base.
(*b*) the altitude on the base and one of the equal sides.
(*c*) the vertical angle and the altitude drawn through this angle.

5. To construct a right triangle, given:
(*a*) The hypotenuse and one leg.
(*b*) One leg and the adjacent acute angle.
(*c*) The hypotenuse and one acute angle.
(*d*) One leg and the altitude on the hypotenuse.



6. To construct an isosceles triangle, given the sum of the sides (EF) and the altitude (AD). Suppose the problem solved, and that ABC is the required triangle. Then it is clear that $EB = BA$, and $FC = CA$. It follows that AD bisects EF. To find B and C, join A to E and to F and draw perpendicular bisectors of AE and AF.

223. Problem. *To construct the third angle of a triangle when two of the angles are given.*

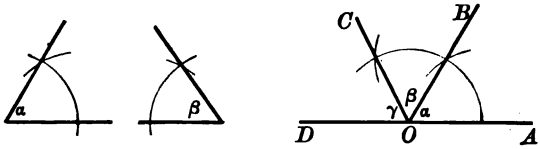


Fig. 93.

Given the angles α, β of a triangle.

Required to construct the third angle γ.

Construction. On a half line OA construct adjacent angles AOB and BOC equal to α and β, respectively.

Produce AO to D.

Then $\angle COD$ is the angle γ required.

Proof. (Let the student give it.)

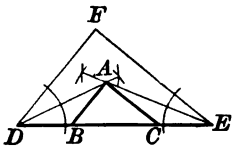
Exercises. 1. To construct a triangle, given two angles and a side opposite one of them. Find the third angle by § 223 and apply § 222.

2. To construct a right triangle, given one acute angle and the opposite leg.

3. To construct a triangle given two sides and the angle opposite one of them.

Discussion. Under what conditions would no solution be possible? One solution? Two solutions?

4. To construct an isosceles triangle, given the base and the opposite angle.



5. To construct a triangle, given the sum of the sides and two angles. Let DE be the sum of the sides, and FDE and FED two angles.

CHAPTER IV

QUADRILATERALS AND POLYGONS IN GENERAL

224. A polygon. If a broken line (§ 13) forms a closed figure, that is, if the two end points of the broken line coincide, the figure is called a *polygon.*

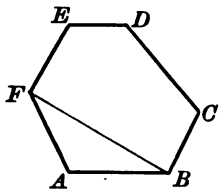


FIG. 94.

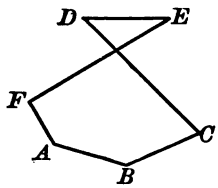


FIG. 95.

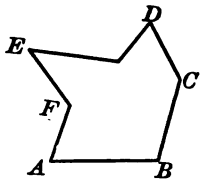


FIG. 96.

In the polygons *ABCDEF*, the points *A, B, C, D, E, F* are called the *vertices*, the segments *AB, BC, CD, DE, EF, FA* are called the *sides*, and the angles *ABC, BCD, CDE, DEF, EFA, FAB*, each formed by two consecutive sides, are called *the angles* of the polygons.

Adjacent angles of a polygon are two angles that have a side in common; *adjacent sides* are two sides that have a vertex in common.

The sum of the sides of a polygon is called the *perimeter;* and the segments joining non-consecutive vertices as (*BF*, Fig. 94), are called *diagonals.*

A polygon is said to be *convex* (Fig. 94) if for each one of the straight lines passing through two consecutive vertices all the other vertices are on the same side of that line. (Figures 95, 96 are polygons which are not convex. Why?)

NOTE. In elementary geometry only convex polygons are considered. In the sequel, therefore, the word "polygon" refers to "convex polygon," unless the contrary is explicitly stated.

225. FUNDAMENTAL PROPOSITION. *A straight line which passes through a point on a convex polygon, not a vertex, cuts the polygon in one other point, and no more.*

NOTE. The last part of this proposition is a consequence of the definition of a convex polygon.

226. A quadrilateral. A polygon which has four sides is called a *quadrilateral.*

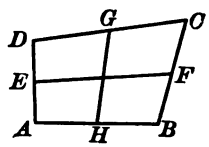


FIG. 97.

Opposite sides of a quadrilateral are two sides which do not have a vertex in common; *opposite angles* are two angles which have no common side. Thus (Fig. 97), AD and CB are opposite sides; $\angle A$ and $\angle C$ are opposite angles. It will be noted that a quadrilateral has two diagonals.

The segments joining the mid-points of the opposite sides of a quadrilateral are called the *medians.* Thus (Fig. 97), EF and GH are the medians of the quadrilateral $ABCD$.

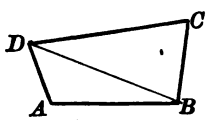


FIG. 98.

227. THEOREM. *The sum of the angles of a quadrilateral is equal to four right angles.*

(The proof is left to the student. See § 180.)

228. A trapezoid. If two sides of a quadrilateral are parallel, the quadrilateral is called a *trapezoid.* The parallel sides are called the *bases,* and the other two sides are called the *legs* of the trapezoid.

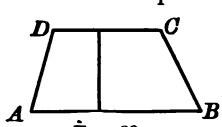


FIG. 99.

The perpendicular distance between the bases of a trapezoid is called the *altitude* of the trapezoid.

A trapezoid with equal legs is called an *isosceles trapezoid.*

229. A parallelogram. If a quadrilateral has two pairs of parallel sides, it is called a *parallelogram* (▱). Hence, a parallelogram is also a trapezoid with parallel legs.

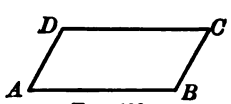


FIG. 100.

230. THEOREM. *A parallelogram is symmetric with respect to the mid-point of either diagonal.*

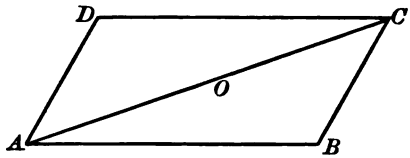


FIG. 101.

Given the ▱ $ABCD$ with O the mid-point of the diagonal AC.

To prove that the ▱ is symmetric with respect to O as a center.

Proof. 1. Imagine the figure to rotate about O until A and C exchange places.

2. Then AB will exchange places with CD and CB will exchange places with AD.
For $\angle DCA = \angle BAC$, and $\angle ACB = \angle CAD$. Why?

3. $\therefore$ B and D will exchange places. Why?

4. $\therefore$ the ▱ is symmetric with respect to O as a center. Def., § 132

231. COROLLARY 1. *Either diagonal divides a parallelogram into two congruent triangles.*

232. COROLLARY 2. *The opposite sides of a parallelogram are equal.*

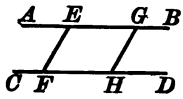


FIG. 102.

233. COROLLARY 3. *Parallel segments cut off between parallel lines are equal.*

234. COROLLARY 4. *The diagonals and the medians of a parallelogram bisect each other.*

235. COROLLARY 5. *Any two opposite angles of a parallelogram are equal, and any two adjacent angles are supplementary.*

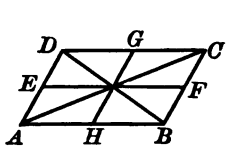


FIG. 103.

236. COROLLARY 6. *If two adjacent sides of a parallelogram are equal, all the sides are equal.* § 232

237. COROLLARY 7. *If one angle of a parallelogram is a right angle, all the angles are right angles.* § 235

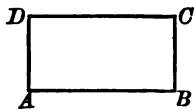


FIG. 104.

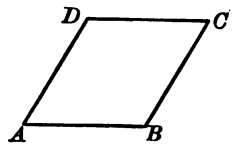


FIG. 105.

238. A rectangle. If one angle of a parallelogram is a right angle, the figure is called a *rectangle* (Fig. 104).

239. A rhombus. If two adjacent sides of a parallelogram are equal, the figure is called a *rhombus* (Fig. 105).

240. THEOREM. *All the angles of a rectangle are right angles.* §§ 237, 238

241. THEOREM. *All the sides of a rhombus are equal.* §§ 236, 239

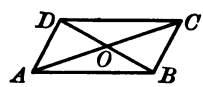
242. A square. If two adjacent sides of a rectangle are equal, the figure is called a *square.*

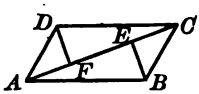
Hence, a square is also a rhombus which has a right angle.

243. **EXERCISES**

1. The diagonals of a parallelogram bisect each other. Prove by congruent triangles.

$\triangle AOB \equiv \triangle COD.$

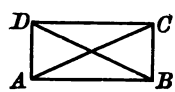




2. In the parallelogram $ABCD$, BE and DF are perpendicular to the diagonal AC. Prove that $BE = DF$.

3. The diagonals of a rectangle are equal.

$\triangle BAD \equiv \triangle ABC.$



244. THEOREM. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

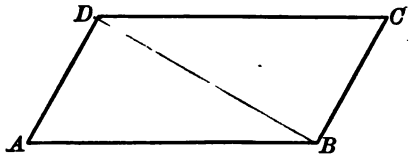


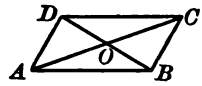
FIG. 106.

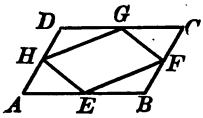
(The proof is left to the student.)

SUGGESTION: Draw a diagonal BD. $\triangle ABD \equiv \triangle CBD$. Show that the opposite sides are parallel by § 162.

245. EXERCISES

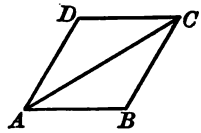
1. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

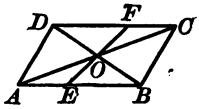




2. If the mid-points of the sides of a parallelogram are joined in order, the resulting figure is also a parallelogram. (This may be proved in at least two ways.)

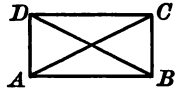
3. If either diagonal of a parallelogram bisects one of the angles, the figure is a rhombus.





4. In the parallelogram $ABCD$ the diagonals intersect in O. The segment EF passes through O and has its end-points on AB and DC. Prove that $OE = OF$.

5. If the diagonals of a parallelogram are equal, the figure is a rectangle.



$\triangle BAD \equiv \triangle ABC$. $\therefore \angle DAB = \angle CBA$.

But $\angle DAB + \angle CBA = 180°$. § 235

246. THEOREM. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*

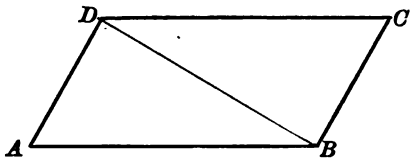


FIG. 107.

(The proof is left to the student.)

247. COROLLARY 1. *The median drawn to two sides of a parallelogram is parallel to the other two sides, and is equal to each of them.*

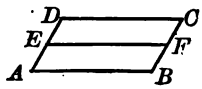
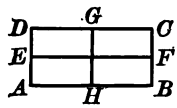


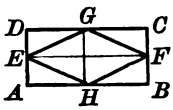
FIG. 108.

248. EXERCISES

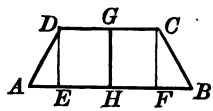


1. Either median of a rectangle is an axis of symmetry of the figure.

2. The diagonals of a rectangle are equal. Prove by symmetry.



3. The segments joining the mid-points of the sides of a rectangle, taken in order, form a rhombus. Prove by symmetry.



4. The median drawn to the bases of an isosceles trapezoid (not a parallelogram) is an axis of symmetry of the figure.

5. Each base of an isosceles trapezoid (not a parallelogram) makes equal angles with the legs.

6. The diagonals of an isosceles trapezoid (not a parallelogram) are equal. Prove by symmetry.

7. Prove Ex. 6 by congruent triangles.

8. The segments joining a pair of opposite vertices to the mid-points of a pair of opposite sides form with the sides a parallelogram.

249. THEOREM. *The median drawn to the legs of a trapezoid is parallel to the bases and is equal to half their sum.*

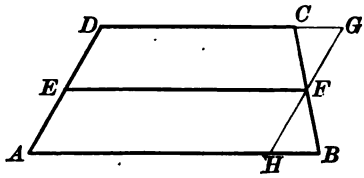


FIG. 109.

Given the trapezoid $ABCD$ with the median EF drawn to the legs AD and BC.

To prove that $EF \parallel AB$ and DC, and that $EF = \frac{1}{2}(AB + DC)$.

Proof. 1. Draw HG through F parallel to AD, cutting AB and DC, produced if necessary, in H and G, respectively.

2. Then $\triangle HBF \equiv \triangle GCF$. Why?

3. $\therefore FG = FH$,
and $CG = HB$. Why?

4. $\therefore FH = EA$. Why?

5. $\therefore EF \parallel AH$ and DG. §§ 246, 81
That is, $EF \parallel AB$ and DC.

6. Also $EF = DG = DC + CG$. § 232
And $EF = AH = AB - HB$.

7. $\therefore$ observing that $HB = CG$,
$EF = DC + CG$,
and $EF = AB - CG$.

8. $\therefore$ by addition,
$2\,EF = AB + DC$,

9. Or, $EF = \dfrac{AB + DC}{2}$

250. COROLLARY. *If a straight line is parallel to a base of a trapezoid and bisects one leg, it bisects the other leg also.*

For a straight line drawn through E parallel to AB coincides with EF (§ 158).

251. THEOREM. *If three or more parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.*

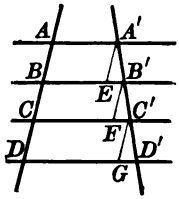


FIG. 110.

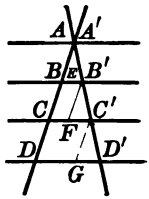


FIG. 111.

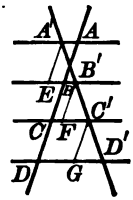


FIG. 112.

Given three or more parallel lines AA', BB', CC', DD', cutting off equal segments $AB = BC = CD$ on the transversal AD, and any other transversal $A'D'$, meeting the given lines in A', B', C', D'.

To prove that the segments $A'B'$, $B'C'$, $C'D'$ are equal.

Proof. Three cases must be considered, according as the transversals intersect beyond the parallel lines (Fig. 110), on one of them (Fig. 111), or between two of them (Fig. 112). The following proof applies to each case.

(1) Through A', B', C' draw lines parallel to AD and meeting BB', CC', DD' in E, F, G, respectively. (In Fig. 111, E coincides with B.)

(2) Then $AB = A'E$, $BC = B'F$, $CD = C'G$. § 233

(3) $\therefore A'E = B'F = C'G$. Why?

(4) $\therefore \triangle A'EB' \equiv \triangle B'FC' \equiv \triangle C'GD'$. Why?

(5) $\therefore A'B' = B'C' = C'D'$. Why?

252. COROLLARY. *A straight line parallel to one side of a triangle which bisects one of the other two sides bisects the third side also.*

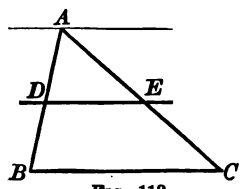


FIG. 113.

253. THEOREM. *The segment joining the mid-points of two sides of a triangle is parallel to, and equal to one half of, the third side of the triangle.*

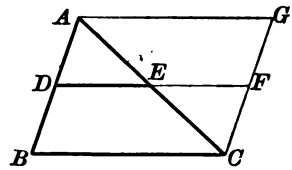


FIG. 114.

Let ABC be the triangle and ED the given segment. Complete the parallelogram $ABCG$ (Fig. 114).

(Let the student complete the proof.)

254. THEOREM. *The medians of a triangle meet in a point which is two thirds of the distance from each vertex to the mid-point of the opposite side.*

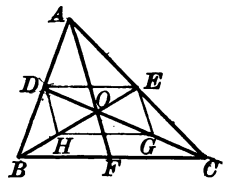


FIG. 115.

Given the triangle ABC with medians BE and CD meeting in O.

To prove that the third median AF passes through O and that $BO = \frac{2}{3} BE$, $CO = \frac{2}{3} CD$, and $AO = \frac{2}{3} AF$.

Proof. 1. Let H and G be the mid-points of BO and CO, respectively, and join ED, DH, HG, and GE.

2. Then $DE \parallel BC \parallel HG$, and $DE = \frac{1}{2} BC = HG$. § 253

3. $\therefore DE \parallel HG$ and $DE = HG$.

4. $\therefore DEGH$ is a parallelogram. Why?

5. $\therefore OH = OE$ and $OG = OD$. Why?

6. $\therefore BO = \frac{2}{3} BE$ and $CO = \frac{2}{3} CD$. Why?

7. $\therefore$ the median AF also passes through O and $AO = \frac{2}{3} AF$. Why?

255. THEOREM. *If the sides of a quadrilateral on one side of a diagonal are equal, respectively, to the adjacent sides on the other side of the diagonal, this diagonal is an axis of symmetry of the figure.*

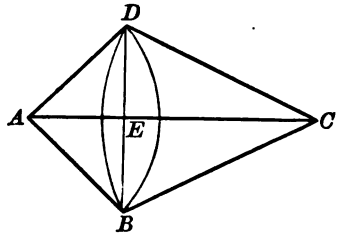


FIG. 116.

(The proof is left to the student.)

SUGGESTION: Circles may be described from A and C as centers, cutting each other in B and D.

256. A kite. A quadrilateral which has a diagonal as an axis of symmetry is called a *kite*. The other diagonal is called the *transverse axis*.

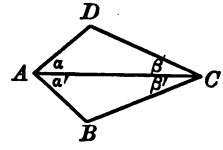
257. COROLLARY 1. *The axis of a kite is the perpendicular bisector of the other diagonal.*

258. COROLLARY 2. *The axis of a kite bisects the angles whose vertices it joins.*

259. COROLLARY 3. *The axis of a kite divides it into two congruent triangles.*

260. COROLLARY 4. *The diagonals of a rhombus are perpendicular and bisect the angles of the rhombus.*

For a rhombus is a kite with equal sides.



Exercise. If one diagonal of a quadrilateral bisects the angles at the vertices which it joins, the figure is a kite.

261. THEOREM. *Two quadrilaterals are congruent, if three sides and the included angles of the one are equal, respectively, to three sides and the included angles of the other.*

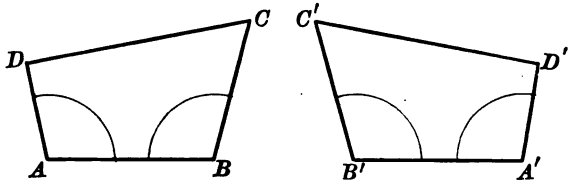


FIG. 117.

Given the quadrilaterals $ABCD$ and $A'B'C'D'$ with $DA = D'A'$, $AB = A'B'$, $BC = B'C'$, $\angle A = \angle A'$, and $\angle B = \angle B'$.

To prove that quadrilateral $ABCD \equiv$ quadrilateral $A'B'C'D'$.

Proof. 1. Imagine the quadrilateral $A'B'C'D'$ to slide along the line BB' until B' coincides with B; then to rotate about B as a center until $B'A'$ coincides with its equal BA. If then the quadrilaterals are not on the same side of AB, turn the quadrilateral $A'B'C'D'$ over on AB as an axis.

2. Then the straight line $A'D'$ will coincide with the straight line AD. Why?

3. The point D' will coincide with the point D. Why?

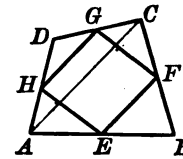
(Let the student complete the proof.)

262. COROLLARY 1. *Two parallelograms are congruent if two adjacent sides and the included angle of the one are equal, respectively, to two adjacent sides and the included angle of the other.*

263. COROLLARY 2. *Two rectangles are congruent if two adjacent sides of the one are equal, respectively, to two adjacent sides of the other.*

264. COROLLARY 3. *Two squares are congruent if a side of the one is equal to a side of the other.*

265. EXERCISES

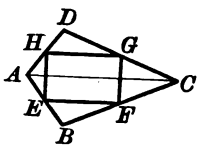


1. The segments joining the mid-points of the sides of a quadrilateral, taken in order, form a parallelogram.

Draw a diagonal AC.

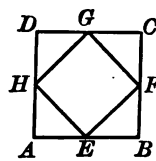
HG and EF are each parallel to AC, and equal to $\frac{1}{2}$ AC. § 253

2. The medians of a quadrilateral bisect each other.



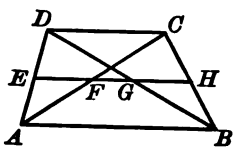
3. The segments joining the mid-points of the sides of a kite, taken in order, form a rectangle.

4. The medians of a kite are equal.



5. The segments joining the mid-point of the sides of a square, taken in order, form a square. Cf. Ex. 3, above, and Ex. 3, § 248.

6. The median drawn to the legs of a trapezoid bisects the diagonals. §§ 249, 253



7. The segment joining the mid-points of the diagonals of a trapezoid is equal to half the difference of the bases.

$$EF + FG = \tfrac{1}{2} \cdot AB$$

and

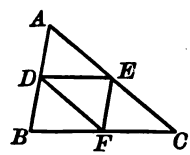
$$GH + FG = \tfrac{1}{2} \cdot AB$$

$$\therefore EF + GH + 2\,FG = AB$$

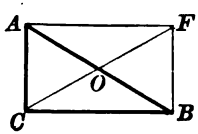
But

$$EF = \tfrac{1}{2} \cdot DC \text{ and } GH = \tfrac{1}{2} \cdot DC.$$

$$\therefore DC + 2\,FG = AB, \text{ or } 2\,FG = AB - DC.$$

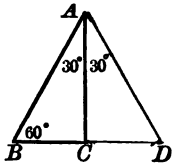


8. The segments joining the mid-points of the sides of a triangle divide the triangle into four congruent triangles.



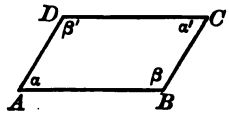
9. The median drawn to the hypotenuse of a right triangle is equal to half the hypotenuse. Compare with Ex. 13, § 209.

Construct the rectangle $CBFA$.



10. If one acute angle of a right triangle is 60°, the hypotenuse is double the shorter leg.

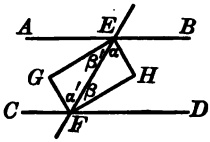
Draw AD, making $\angle CAD = 30°$. Then $\triangle ABD$ is equilateral.



11. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

$$\alpha + \alpha' + \beta + \beta' = 4\,R \qquad \S\ 227$$

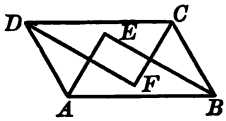
$\therefore 2\,\alpha + 2\,\beta = 4\,R$, or $\alpha + \beta = 2\,R$. $\therefore AD \parallel BC$, etc.



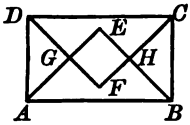
12. If two parallel lines are cut by a transversal, the bisectors of the interior angles form a rectangle.

$\alpha = \alpha'$ and $\beta = \beta'$. $\therefore EH \parallel GF$ and $FH \parallel GE$. Also $\alpha + \beta' = 90°$.

13. The bisectors of two adjacent angles of a parallelogram are perpendicular, and the bisectors of two opposite angles are parallel.

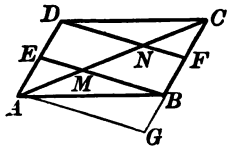


14. The bisectors of the angles of a parallelogram form a rectangle.

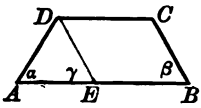


15. The bisectors of the angles of a rectangle form a square. Prove by symmetry.

16. Line segments drawn from two opposite vertices of a parallelogram to the mid-points of two opposite sides divide the diagonal joining the other two vertices into three equal parts.



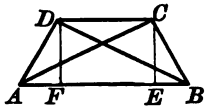
Draw $AG \parallel EB$ meeting CB produced in G. Show that $DF \parallel EB \parallel AG$; also that $BG = BF = FC$. Then $AM = MN = NC$ (§ 251).



17. If a base makes equal angles with the legs of a trapezoid, the trapezoid is isosceles.

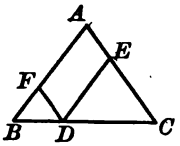
Draw $DE \parallel CB$, and show that $\triangle AED$ is isosceles. $\therefore DA = DE = CB$. (Why?)

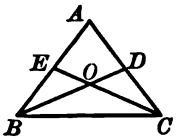
18. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.



Draw CE and $DF \perp AB$. Rt. $\triangle AEC \equiv$ Rt. $\triangle BFD$. Then $\triangle CAB \equiv \triangle DBA$.

19. If from any point on the base of an isosceles triangle parallels to the equal sides are drawn, a parallelogram is formed whose perimeter is equal to the sum of the equal sides of the triangle.

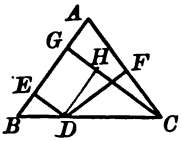




20. If the medians to two sides of a triangle are equal, the triangle is isosceles.

$BO = CO$ (§ 254). $\therefore \angle OBC = \angle OCB$.
$\triangle DBC \equiv \triangle ECB$.

21. The sum of the perpendiculars from any point on the base of an isosceles triangle to the equal sides of the triangle is equal to the altitude upon one of the equal sides.



Draw $DH \perp CG$. $ED = GH$. It remains to prove that $DF = CH$. Show that $\triangle DHC \equiv \triangle CFD$.

POLYGONS IN GENERAL

266. A polygon is called a *triangle*, *quadrilateral*, *pentagon*, *hexagon*, *heptagon*, *octagon*, *nonagon*, *decagon*, ···, *dodecagon*, ···, *pentadecagon*, ···, *n-gon*, according as it has 3, 4, 5, 6, 7, 8, 9, 10, ···, 12, ···, 15, ···, n sides.

267. An equilateral polygon. If the sides of a polygon are equal, it is called an *equilateral* polygon.

268. An equiangular polygon. If the angles of a polygon are equal, it is called an *equiangular* polygon.

269. A point is said to be *within* a convex polygon if a straight line drawn through the point, in the plane of the polygon, cuts the polygon in two points on opposite sides of the given point.

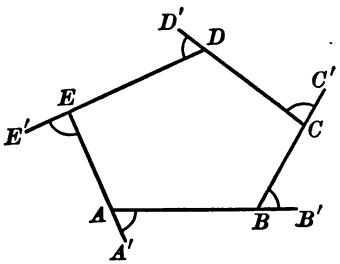


FIG. 118.

270. An exterior angle of a polygon. The angle formed by one side of a polygon and an adjacent side produced through the common vertex is called an *exterior angle* of the polygon.

271. THEOREM. *The sum of the angles of a convex polygon of n sides is* $(n-2) \cdot 2\,R$.

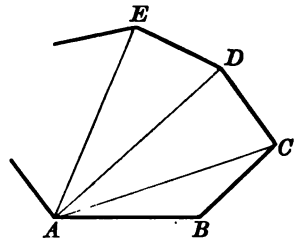


FIG. 119.

Given the polygon $ABCDE \cdots$ having n sides.

To prove that the sum of the angles of the polygon is $(n-2) \cdot 2\,R$.

Proof. 1. Imagine all the diagonals drawn from the vertex A.

2. The number of triangles thus formed is $n-2$ (for to each side, save two, there corresponds one triangle).

3. But the sum of the angles of each triangle is $2\,R$. § 180

4. $\therefore$ the sum of the angles of the polygon is

$$(n-2)\,2\,R,$$

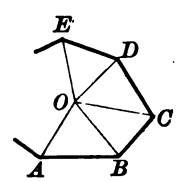
(for the sum of the angles of the polygon is equal to the sum of the angles of the triangles).

272. COROLLARY 1. *Each angle of an equiangular polygon having n sides is* $\dfrac{(n-2)\,2\,R}{n}$.

Exercises. 1. Prove § 271 by joining any point O in the polygon to the n vertices.

2. In a polygon having n sides how many diagonals can be drawn from each vertex? How many can be drawn from all the vertices taken together?

3. How many degrees in each angle of an equiangular polygon of 512 sides?



273. THEOREM. *The sum of the exterior angles of a convex polygon, formed by producing the sides successively in the same sense, is* 4 *R*.

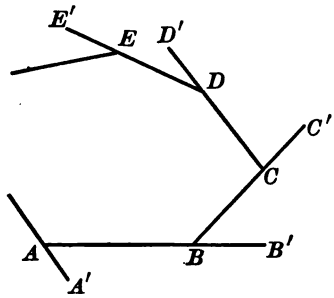


FIG. 120.

Given a polygon $ABCDE\cdots$ with the sides produced successively in the same sense, forming the exterior angles $A'AB$, $B'BC$, $C'CD$, $\cdots$.

To prove that the sum of the exterior angles is $4\,R$.

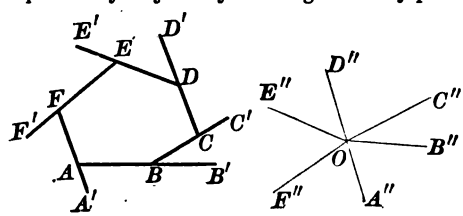
Proof. 1. The sum of the exterior and interior angles formed at each vertex is $2\,R$. Why?

2. $\therefore$ the sum of the interior and exterior angles of the polygon is $n \cdot 2\,R$, where n is the number of sides. Why?

3. But the sum of the interior angles of the polygon is $(n-2) \cdot 2\,R$. § 271

4. $\therefore$ the sum of the exterior angles is
$$2\,nR - (n-2) \cdot 2\,R, \text{ or } 4\,R.$$ Why?

Exercise. Prove § 273 independently of § 271 by drawing from any point O half lines parallel to AB, BC, CD $\cdots$, respectively, and showing that the sum of the angles thus formed about the point O is equal to the sum of the exterior angles of the polygon.



274. THEOREM. *If a polygon is symmetric with respect to two axes which are perpendicular to each other, it is symmetric with respect to their intersection as a center.*

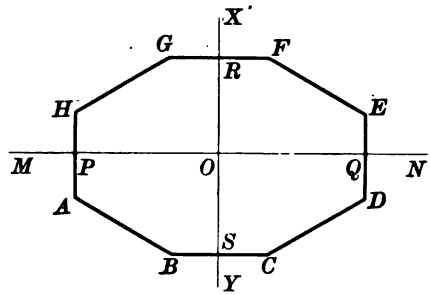


FIG. 121.

Given the polygon $ABCDEFGH$ symmetric with respect to the two perpendicular axes MN and XY, which intersect in O.

To prove that the polygon is symmetric with respect to O as a center.

Proof. 1. The polygons $ABSOP$ and $OQEFR$ are each congruent to the polygons $SCDQO$ and $PORGH$. Why?

2. $\therefore$ $ABSOP \equiv OQEFR$, and $SCDQO \equiv PORGH$. Why?

3. $\therefore$ if the polygon $ABCDEFGH$ is rotated about O as a center until P and Q exchange places, the figure will coincide with its original position. Why?

4. $\therefore$ the polygon $ABCDEFGH$ is symmetric with respect to O as a center.

275. COROLLARY 1. *If a polygon has a center of symmetry, any straight line drawn through the center divides the polygon into two congruent parts.*

276. COROLLARY 2. *If a polygon has an axis of symmetry, any diagonal joining corresponding vertices is bisected by the axis, and is perpendicular to it.*

277. COROLLARY 3. *If a polygon has a center of symmetry, any diagonal joining corresponding vertices passes through the center of symmetry and is bisected by it.*

278. **EXERCISES**

1. Find the sum of the angles of a pentagon; of a hexagon; of a heptagon; of a polygon of 64 sides.

2. The sum of the angles of a polygon is $12\,R$. Find the number of sides.

Let n be the number of sides. Then $(n-2)\cdot 2\,R = 12\,R$. Solve for n.

3. Each angle of an equiangular polygon is 150°. Find the number of sides.

4. How many straight lines are determined, in general, by n points taken at random?

5. Four angles of a pentagon are 100°, 90°, 75°, 125°. Find the remaining angle.

6. Find the number of degrees in each exterior angle of an equiangular octagon.

7. If two angles of a quadrilateral are right angles, the other two angles are supplementary.

8. If three angles of a quadrilateral are right angles, the quadrilateral is a rectangle.

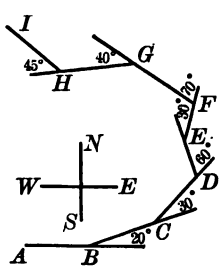
9. If three angles of a pentagon are right angles, find the sum of the other two angles.

10. If three angles of a quadrilateral are each three times the remaining angle, find the angles.

11. In what direction must a ball on a billiard table be struck in order that it may return to its original position, assuming that the directions in which it strikes and leaves a cushion make equal angles with the cushion?

12. If a polygon has two axes of symmetry which are perpendicular to each other, every point on the polygon corresponds to three other points which form a rectangle, and every straight line joining two points on the polygon corresponds to three other straight lines which form a rhombus.

13. The figure shows a drawing of a preliminary survey for a branch railway. If the direction of AB is due east, find the direction of HI (that is, the angle which HI makes with a north-and-south line drawn through H).



CONSTRUCTION PROBLEMS

279. PROBLEM. *To construct a parallelogram, given two adjacent sides and the included angle.*

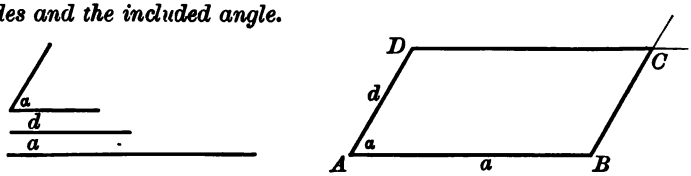


FIG. 122.

Given the sides a, d, and the included angle α.

Required to construct a parallelogram.

Construction. Draw $AB = a$. At A construct $\angle BAD = \alpha$, and lay off $AD = d$.

From B and D draw parallels to AD and AB, respectively. These lines will intersect in a point C.

Then $ABCD$ is a parallelogram satisfying the required conditions.

Proof. 1. $AB = a$, $AD = d$, and $\angle BAD = \alpha$. Construction

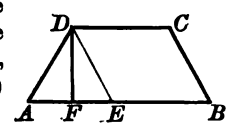
2. Also $BC \parallel AD$, and $DC \parallel AB$. Construction

3. $\therefore$ $ABCD$ is a parallelogram. Def.

Exercises. To construct a parallelogram:

1. Given two adjacent sides and a diagonal.
2. Given one side and the two diagonals.
3. Given the diagonals and one of the angles formed by them.
4. Given one side, one angle, and one diagonal.
5. Given one side, the altitude on that side, and one angle.
6. To construct a square, given a diagonal.
7. To construct a rhombus, given the two diagonals.
8. To construct a rhombus, given one angle and one diagonal.
9. To construct a rectangle, given one side and the angles formed by the diagonals.
10. To construct an isosceles trapezoid, given the bases and the altitude.

Assume the problem solved. Let AB and DC be equal to the given bases, and let DF be equal to the given altitude. Draw $DE \parallel CB$. Then $EB = DC$, $\triangle ADE$ is isosceles, and DF bisects AE. (Why?) Now try to work backwards.



280. PROBLEM. *To divide a given line segment into a given number of equal parts.*

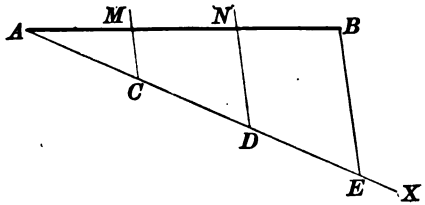


FIG. 123.

Given the line segment AB.

Required to divide AB into any number of equal parts, say three.

Construction. From A draw AX, making any convenient angle with AB.

On AX lay off equal segments AC, CD, DE any convenient length.

Draw EB, and draw CM and DN parallel to EB.

These lines will divide AB into three equal parts.

Proof. 1. $AC = CD = DE$. Why?

2. And CM, DN, EB are parallel. Why?

3. $\therefore$ these lines cut off equal parts on AB. Why?

Discussion. AB may be divided into any given number of equal parts by laying off the given number of equal segments on AX and proceeding as above.

Exercises. 1. To construct an equilateral triangle, given the perimeter.

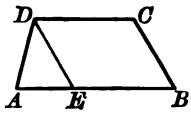
2. To construct a triangle, given one side and the medians drawn from its extremities. See § 254.

3. To construct a triangle, given two sides and the median drawn to the other side.

4. To construct a triangle, given the three medians.

281. MISCELLANEOUS EXERCISES

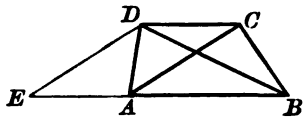
1. To construct an isosceles trapezoid, given the bases and one angle.
2. To construct an isosceles trapezoid, given the bases and one diagonal.



3. To construct a trapezoid, given the four sides.

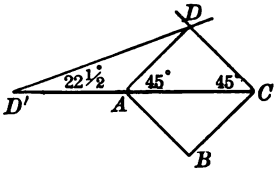
Assume the problem solved. Let AB and DC be equal to the given bases, and let AD and BC be equal to the given legs, respectively. Draw $DE \parallel BC$. Then AE is the difference of the two bases, and $DE = CB$. (Why?) Now try to work backwards.

4. To construct a trapezoid, given the two bases and the two diagonals.



Let $ABCD$ be any trapezoid with bases AB and CD. Draw DE parallel to AC, intersecting BA produced in E. Then EB is the sum of the bases, and ED is equal to the diagonal AC. (Why?)

5. To construct a square, given the sum of one diagonal and one side.



Let $D'C$ be the sum of the diagonal and side. Suppose the problem is solved, and that $ABCD$ is the required square.

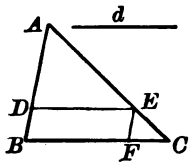
Then $D'A = AD = DC$. (Why?)

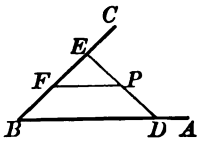
$\therefore \angle DCA = \angle DAC = 45°$,

and $\angle AD'D = \frac{1}{2} \angle DAC = 22\frac{1}{2}°$. (Why?)

6. To construct a rectangle, given a diagonal and the sum of two sides.

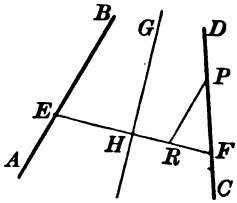
7. To draw through two sides of a triangle a line parallel to the third side, so that the segment intercepted between the sides shall be equal to a given segment.





8. To draw through a given point P between the sides of an angle ABC a line so that the segment between the sides shall be bisected by P.

9. To bisect an angle when the vertex is inaccessible.



Suppose the problem is solved, and that HG is the required bisector. Then HG is an axis of symmetry of the figure. Draw $EF \perp HG$.

$\angle BEF = \angle DFE$. (Why?)

Draw $PR \parallel BA$.

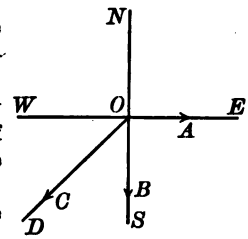
Then $\angle PRF = \angle BEF = \angle DFE$. (Why?)

Now work backwards.

NOTE. *In mechanics, motion in a straight line is represented by a segment, the direction of the line indicating the direction of the motion, and the length of the segment representing the magnitude of the motion.*

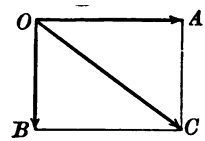
10. Three men start from St. Louis: the first going 8 miles eastward; the second going 12 miles southward; the third going 16 miles southwesterly. Represent these journeys by line segments.

Represent the position of St. Louis by the point O. Through O draw two perpendicular lines WE and NS as shown. The directions E, N, W, S are represented by the sense of the half lines OE, ON, OW, OS, respectively. Choose a convenient scale, say, 1 inch = 4 miles. On OE lay off $OA = 2$ inches, on OS lay off $OB = 3$ inches; draw OD, making an angle of 45° with OS, and lay off $OC = 4$ inches. Then the segments OA, OB, OC will represent the journeys, the arrowheads indicating the directions of the motions.



11. An attempt is made to row a boat at the rate of 4 miles an hour directly across a stream flowing at the rate of 3 miles an hour. Represent the motion of the boat.

We assume the motion of the boat to be the resultant of two independent component motions, one due to the propulsion of the oars and one due to the force of the current. That is, in a given time the resultant of the two component motions acting simultaneously is assumed to be the same as if they acted consecutively, or as if the boat first floated down stream for a given unit of time without propulsion of the oars and was then rowed across through still water for an equal period of time. Therefore, the construction is as follows:



Draw OA 4 units in length. This will represent the component motion in a given time due to the propulsion of the oars. Draw $OB \perp OA$, making OB 3 units in length. This will represent the component motion due to the stream. Complete the parallelogram $OBCA$, and draw the diagonal OC. OC will represent the motion of the boat in the given unit of time.

NOTE. *When two component uniform motions acting along straight lines have different directions, the process of finding the resultant involves what is known as* THE PRINCIPLE OF THE PARALLELOGRAM OF MOTIONS, viz.: If the segments which represent the components are made adjacent sides of a parallelogram, the diagonal drawn from the angle included by these sides will represent the resultant in both magnitude and direction.

12. A flag is drawn steadily downward 60 feet from the masthead of a moving ship. During the same time, the ship moves forward 24 feet. Represent the direction and length of the actual path of the flag.

13. A sailor climbs a mast at the rate of 3 feet a second. The ship is sailing at the rate of 12 feet a second. Represent the path through which he actually moves during 20 seconds.

14. A railway train is moving northeastward at the rate of 30 miles an hour. (*a*) Represent the rate at which it is moving eastward; (*b*) represent the rate at which it is moving northward.

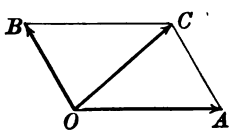
15. A train is moving with a speed of 3 miles an hour. A brakeman on the top of the train runs toward the engine with a speed of 10 feet a second. What is his resultant speed? What would be his resultant speed if he were to run with the same speed toward the rear of the train?

16. If the boatman in Ex. 11 wished to go straight across the stream in spite of the current, represent the direction which he must row.

17. A sledge party is traveling on the ice toward the north pole at the rate of one mile an hour. The ice is drifting southeastward at the rate of 22 yards a minute. Represent the actual path of the party.

18. A football receives, simultaneously, two horizontal blows, — one from the west, having a force of 18 lb., and one from the southeast, having a force of 12 lb. Represent the direction of its motion.

By the second law of motion (see works on physics) the motion imparted by each blow is in the direction of the impressed force and is proportional to it. Hence, construct a parallelogram having adjacent sides 18 units and 12 units with included angle 135°. Then the diagonal drawn from the included angle will represent the direction of the football.



NOTE. Since motion takes place in the direction of the impressed force and is proportional to it, the diagonal OC also represents a single force which is equivalent to the given forces OA and OB. Hence, to find the resultant of two concurrent forces acting in different directions, proceed as in finding the resultant of two component motions.

19. A football receives, simultaneously, three horizontal blows, — one from the north, having a force of 10 lb.; one from the east, having a force of 15 lb.; and one from the southeast, having a force of 25 lb. Represent the resultant of the three forces, and the direction of the motion of the football.

20. A woman pushes on the handle of a carpet sweeper with a force of 5 lb. If the handle makes an angle of 60° with the floor, represent the effective force which acts horizontally to move the sweeper; also the force

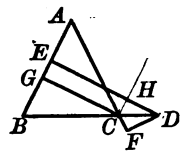
which is exerted perpendicularly on the sweeper. What is the effective force in pounds tending to move the sweeper ?

21. A team of horses pulls a log by a chain inclined at an angle of 30° with the horizontal. If the force exerted on the chain is 500 lb., represent the effective horizontal force, and the vertical force which tends to lift the log. What is the force in pounds which tends to lift the log ?

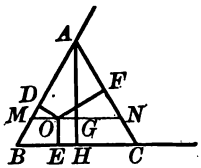
22. Two equal forces acting at an angle of 120° have a resultant of 50 lb. What is the magnitude of each force ?

23. A 200 lb. weight is supported by two strings, each making an angle of 30° with the horizontal. What is the tension of each string ?

24. The difference of the perpendiculars from any point on the base produced of an isosceles triangle to the equal sides of the triangle, or the equal sides produced, is equal to the altitude upon one of the equal sides.



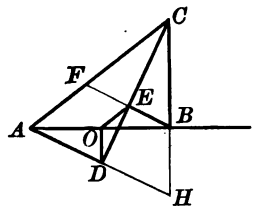
Draw $CH \perp DE$. $CG = HE$. It remains to prove that $DF = DH$



25. The sum of the perpendiculars from any point within an equilateral triangle to the three sides is equal to the altitude.

Draw MN through $O \parallel BC$. $OD + OF = AG$. (Why ?)

26. If perpendiculars are drawn from the extremities of one side of a triangle to the bisector of the opposite angle, the segment joining the mid-point of that side to the foot of either perpendicular is equal to half the difference of the other two sides.



Let BE and AD be $\perp$s on the bisector CO in the triangle ABC. Produce BE to F, and CB to meet AD produced in H.

$\triangle BCE \equiv \triangle FCE$. $\therefore CF = CB$ and $FE = BE$.

$OE = \frac{1}{2} AF$, and $OD = \frac{1}{2} BH$.

CHAPTER V

THE CIRCLE AND THE REGULAR POLYGON. LOCI OF POINTS

282. Extension of the notion of angle. In Chapter I (§ 61) we defined an angle as the *figure* formed by two half lines which issue from the same point. The term "angle" is used also, however, in a somewhat different sense, which may be made clear as follows: When a wheel rotates on its axis, the amount of rotation is conveniently measured by the *angle through which it turns.* Thus, we say "a wheel rotates through an angle of 30° or 60°" and the meaning to be attached to the phrase is evident. Again, in ten minutes, the minute hand of a clock "rotates through an angle of 60°," *i.e.* $\frac{1}{6}$ of a complete revolution. In order to make this use of an angle as a measure of rotation applicable in all cases, however, we must recognize as angles the result of a rotation through more than 180°.

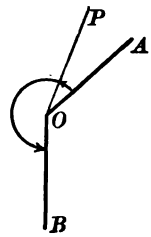


FIG. 124.

Thus, if in the figure (Fig. 124) we suppose the half line OP to have rotated from its *initial position* OA in the direction of the arrowhead to its *terminal position* OB, the angle which measures this rotation is an angle of 230°. (With our previous notion of angle, where we considered only the figure of the two half lines, we should have considered this an angle 130°.)

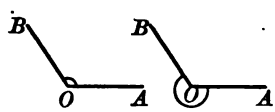
283. Our new notion of angle may be formulated as follows: Imagine a half line OP to rotate about the (fixed) point O from any *initial position*, as OA, to some *terminal position*, as OB. The rotating line is then said to *generate* an angle, and the *size* of the angle is the *amount of rotation* that has taken place. Such an angle is called an *angle of rotation*, when it is necessary to distinguish it from the older more restricted notion.

An angle of rotation may have any size whatever. Thus, if the half line OP makes one complete revolution, it has generated an angle equal to 360° or 4 R. If it continues to rotate in the same sense through another right angle, it will have generated an angle of 450°; and so on.

284. Equal angles of rotation. Two angles of rotation are equal only if they represent the same amount of rotation. Thus, an angle of 90° and one of 450° are not equal, even though the two figures representing the angles are congruent.

285. In particular, if two angles of rotation ABC and $A'B'C'$ are equal and less than two right angles, the angles ABC and $A'B'C'$ are congruent in the older sense of the word angle. Therefore, in discussions where the size of angles is limited to angles that are less than two right angles, the term "angle" may be used in either sense without confusion.

286. Oblique and reflex angles. An angle which is less than two right angles is called an *oblique* angle; an angle which is greater than two right angles but less than four right angles is called a *reflex* angle. An angle which is equal to two right angles is called a *straight* angle.



Oblique Angle Reflex Angle
FIG. 125.

Two half lines issuing from a point O (see Fig. 124) form two angles, one oblique and one reflex, unless the two half lines are in the same straight line. To distinguish between these two angles use is usually made of arcs of circles having their center at O and terminating in the sides of the angle.

ANGLES FORMED BY STRAIGHT LINES WHICH MEET A CIRCLE

287. An inscribed angle. The (oblique) angle formed by two chords which meet on a circle is said to be *inscribed in the circle.* The angle is said to be *subtended* by the arc which lies within the angle and is cut off by the sides.

An angle is said to be *inscribed in an arc* of a circle if its vertex lies on the arc and its sides pass through the extremities of the arc. Thus, the angle ABC is inscribed in the arc ABC and is subtended by the arc ADC.

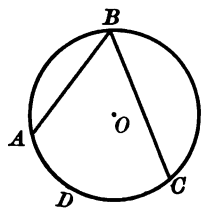


FIG. 126.

288. THEOREM. *An angle inscribed in a circle is equal to half the central angle subtended by the same arc.*

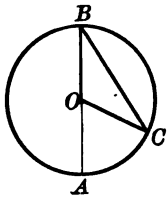


FIG. 127.

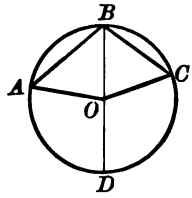


FIG. 128.

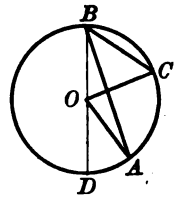


FIG. 129.

Given the angle ABC inscribed in the circle with the center O, and the central angle AOC subtended by the same arc AC.

To prove that $\angle ABC = \frac{1}{2} \angle AOC$.

Proof. We distinguish three cases:

I. When the center O is on a side of the angle ABC.
II. When O is within the angle ABC.
III. When O is without the angle ABC.

Case I.	1. $\angle AOC = \angle OBC + \angle OCB$. (Fig. 127.)	Why?
	2. But $\angle OCB = \angle OBC$.	Why?
	3. $\therefore$ $\angle AOC = 2 \angle OBC$.	Why?
	4. $\therefore$ $\angle ABC = \frac{1}{2} \angle AOC$.	Why?
Case II.	1. Draw the diameter BD (Fig. 128).	
	2. Then $\angle ABD = \frac{1}{2} \angle AOD$.	
	And $\angle OBC = \frac{1}{2} \angle DOC$.	Case I
	3. $\therefore$ $\angle ABC = \frac{1}{2} \angle AOC$.	Why?

Case III. (Fig. 129).

(The proof is left to the student.)

289. COROLLARY 1. *Angles inscribed in the same arc, or in equal arcs, of a circle are equal* (Fig. 130).

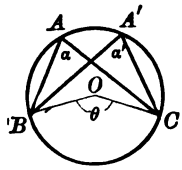


FIG. 130.

290. COROLLARY 2. *An angle inscribed in a semicircle is a right angle* (Fig. 131).

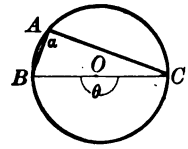


FIG. 131.

291. COROLLARY 3. *The opposite angles of a quadrilateral inscribed in a circle are supplementary* (Fig. 132).

$$\alpha = \tfrac{1}{2}\phi, \text{ and } \gamma = \tfrac{1}{2}\theta. \qquad \text{(Why ?)}$$

$$\therefore \alpha + \gamma = \tfrac{1}{2}(\theta + \phi) = \tfrac{1}{2} \cdot 4R = 2R.$$

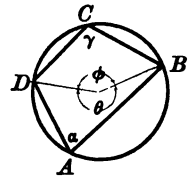
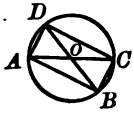


FIG. 132.

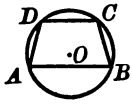
292. **EXERCISES**



1. If the diagonals of a quadrilateral inscribed in a circle are diameters, the quadrilateral is a rectangle.

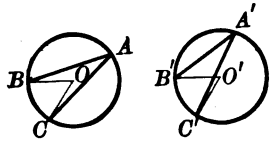
2. If an angle inscribed in a circle is a right angle, the arc in which it is inscribed is a semicircle.

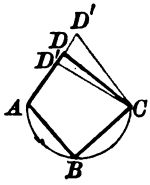
3. Two chords of a circle drawn perpendicular to a third chord at its extremities are equal.



✓4. A trapezoid inscribed in a circle is isosceles.

5. In the same circle, or in congruent circles, equal inscribed angles are subtended by equal arcs.





6. If two opposite angles of a quadrilateral are supplementary, a circle may be circumscribed about the quadrilateral. Prove by reductio ad absurdum:

Let A B and D be supplementary. Describe a circle through A, B, C and suppose it does not pass through D, but cuts AD, or AD produced, in D'.

293. THEOREM. *An angle formed by a tangent and a chord of a circle is equal to any inscribed angle subtended by the arc intercepted between the sides of the angle.*

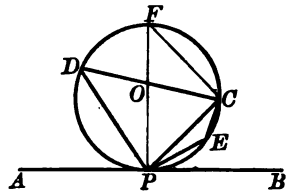


FIG. 133.

Given the circle with the center O, the tangent AB at P, and the chord PC; also the inscribed angles PDC and PEC, subtended by the arcs PEC and PDC, respectively.

To prove that $\angle BPC = \angle PDC$, and $\angle CPA = \angle CEP$.

Proof. 1. Draw the diameter PF, and join FC.

2. Then $\angle$s BPF and FCP are rt. $\angle$s — Why?

3. $\therefore$ $\angle$s BPC and PFC are complements of $\angle CPF$ — Why?

4. $\therefore \angle BPC = \angle PFC$. — Why?

5. $\therefore \angle BPC = \angle PDC$. — Why?

6. Also $\angle CPA$ is supplementary to $\angle BPC$. — Why?
And $\angle CEP$ is supplementary to $\angle PDC$. — Why?

7. $\therefore \angle CPA = \angle CEP$. — Why?

294. COROLLARY 1. *An angle formed by a tangent and a chord of a circle is equal to half the central angle subtended by the arc intercepted between the sides of the angle.*

295. COROLLARY 2. *The tangents to a circle at the extremities of a chord make equal angles with the chord.*

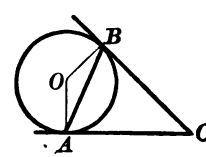


FIG. 134.

296. THEOREM. *The oblique angle included by two half lines which meet a circle is equal to half the sum, or half the difference, of the central angles subtended by the arcs intercepted by the sides (produced, if necessary) according as the vertex of the angle is within or without the circle.*

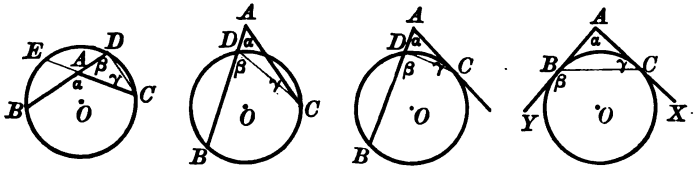


FIG. 135. FIG. 136. FIG. 137. FIG. 138.

(The proof is left to the student.)

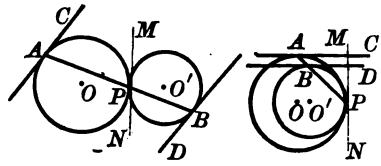
SUGGESTIONS: In Fig. 135, $\alpha = \beta + \gamma$; in Figs. 136, 137, 138, $\alpha = \beta - \gamma$. But β and γ are one half the central angles subtended by the intercepted arcs.

297. EXERCISES

1. If an arc of a circle contains 60°, find the angles formed by the tangents drawn at its extremities; also the angles formed by the tangents with the chord joining the extremities of the arc.

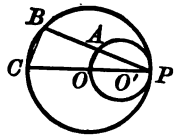
2. If two tangents to a circle form an angle of 135°, find the number of degrees in the intercepted arcs.

3. AB, BC, CD, DE are consecutive arcs of a circle each containing 15°; also AP and BP are tangeants meeting in P. Find the size of $\measuredangle PAB$, PAC, PAD, PAE.

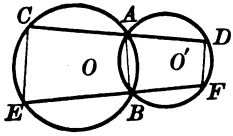


4. If two circles are tangent to each other and a straight line is drawn through the point of contact meeting the circles again in A and B, the tangents to the circles at A and B are parallel.

5. If two circles are tangent to each other and two straight lines are drawn through the point of contact meeting one of the circles in A and B and the other in A' and B', then the cords AB and $A'B'$ are parallel. HINT: Draw a common tangent through the point of contact and apply § 293.

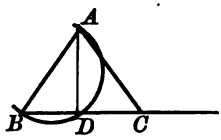


6. If a circle is described on the radius of another circle as a diameter, any chord in the latter drawn from the point in which the circles meet is bisected by the former.



7. If two circles intersect and a line is drawn through each point of intersection and terminated by the circles, the chords joining the ends of these segments are parallel. Draw AB. Then $\angle E$ is supplementary to $\angle CAB$. Why? $\therefore \angle E = \angle BAD$. Why? In like manner it follows that $\angle F = \angle CAB$. $\therefore$ $\angle E$ and F are supplementary. Why?

8. If two circles intersect, any two parallel lines drawn through the points of intersection and terminating in the circles are equal.

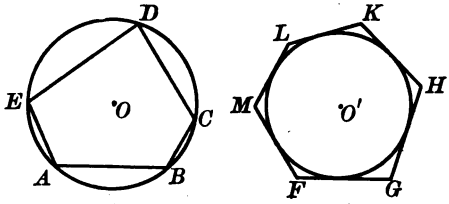


9. A circle is described on one of the equal sides of an isosceles triangle as a diameter. Prove that the circle bisects the base.

REGULAR POLYGONS

298. A regular polygon. If a polygon is both equilateral and equiangular, it is called a *regular polygon.*

The equilateral triangle and the square are examples.



(*a*) FIG. 139. (*b*)

299. Inscribed and circumscribed polygons. If the vertices of a polygon lie on a circle (Fig. 139, (*a*)), the polygon is said to be *inscribed in the circle,* and the circle is said to be *circumscribed about the polygon.*

If the sides of a polygon are tangent to a circle (Fig. 139, (*b*)), the polygon is said to be *circumscribed about the circle,* and the circle is said to be *inscribed in the polygon.*

300. THEOREM. *If a circle is divided into three or more equal arcs, the cords joining the successive points of division form a regular inscribed polygon; and the polygon whose sides are successively tangent to the circle at these points of division is a regular circumscribed polygon.*

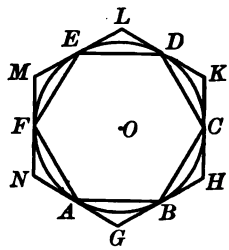


FIG. 140.

Given the circle with center O, divided into equal arcs AB, BC, CD ···; the chords AB, BC, CD ··· forming the inscribed polygon $ABCD$ ···; and tangents NG, GH, HK ··· at points A, B, C, ···respectively, forming the circumscribed polygon $GHKL$ ....

To prove that $ABCD$ ··· and $GHKL$ ··· are regular polygons.

Proof. 1. Imagine the figure to rotate about O until A coincides with the original position of B.

2. The figure will then coincide with its original position. Why?

3. Therefore the polygons are equilateral and equiangular. Why?

4. $\therefore$ $ABCD$ ··· and $GHKL$ ··· are regular polygons. Why?

301. COROLLARY 1. *Two regular polygons of the same number of sides inscribed in the same circle, or in congruent circles, are congruent; and two regular polygons of the same number of sides circumscribed about the same circle, or congruent circles, are congruent.*

302. COROLLARY 2. *If the vertices of a regular polygon inscribed in a circle are joined to the middle points of the arcs subtended by the sides of the polygon, these segments form an inscribed regular polygon of double the number of sides.*

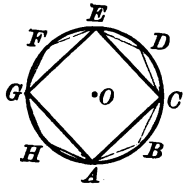


FIG. 141.

303. COROLLARY 3. *If tangents are drawn to a circle at the middle points of the arcs between adjacent points of contact of the sides of a regular polygon circumscribed about the circle, a circumscribed regular polygon of double the number of sides is formed.*

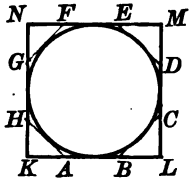


FIG. 142.

304. COROLLARY 4. *The perimeter of a regular polygon inscribed in a circle is less than the perimeter of an inscribed regular polygon of double the number of sides; and the perimeter of a circumscribed regular polygon is greater than the perimeter of a circumscribed regular polygon of double the number of sides.*

305. THEOREM. *The perimeter of a polygon circumscribed about a circle is greater than the perimeter of any polygon inscribed in the same circle.*

(The proof is left to the student.)

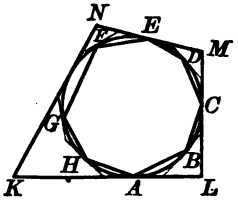


FIG. 143.

306. THEOREM. *A regular polygon is symmetric with respect to the perpendicular bisector of any side.*

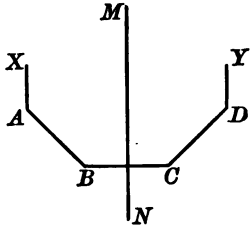


FIG. 144.

(The proof is left to the student.)

307. THEOREM. *A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.*

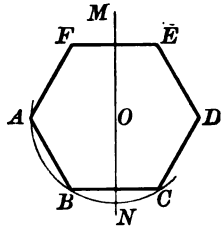


FIG. 145.

Given the regular polygon $ABCDE \cdots$.

To prove (1) that a circle may be circumscribed about $ABCDE \cdots$.

(2) that a circle may be inscribed in $ABCDE \cdots$.

Proof (1). 1. Describe a circle through the vertices A, B, C, and draw MN, the perpendicular bisector of BC. § 89

2. Then the polygon is symmetric with respect to MN, and A and D are corresponding points. § 306

3. $\therefore$ the circle must pass through D. Why?

In like manner it can be proved that the circle must pass through the remaining vertices.

Proof (2). 1. Let O be the center of the circumscribed circle.

2. Imagine the figure to rotate about O until A coincides with the original position of B.

3. The polygon will then coincide with its original position. Why?

4. $\therefore$ O is equally distant from the sides. Why?

5. $\therefore$ A circle may be inscribed in $ACBDE \cdots$. Why?

308. COROLLARY. *The circle circumscribed about, and the circle inscribed in, any regular polygon have a common center.*

309. The center, radius, and apothem of a regular polygon. The common center of the circles, circumscribed about, and inscribed in, a regular polygon is called the *center* of the regular polygon.

The radii of the circles circumscribed about, and inscribed in, a regular polygon are called the *radius* of the regular polygon, and the *apothem*, respectively.

310. An angle at the center of a regular polygon. The angle formed by two radii of a regular polygon drawn to adjacent vertices is called an *angle at the center* of the regular polygon.

311. THEOREM. *An angle at the center of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

(The proof is left to the student.)

312. COROLLARY 1. *The radii of a regular polygon form with the sides as many congruent isosceles triangles as the polygon has sides.*

313. COROLLARY 2. *The radii of a regular polygon bisect the angles of the polygon.*

314. COROLLARY 3. *A side of a regular hexagon is equal to the radius.*

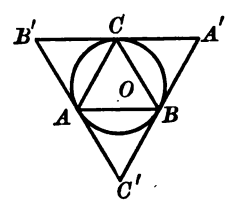
315. EXERCISES

1. If the diagonals of a quadrilateral inscribed in a circle are perpendicular diameters, the quadrilateral is a square.

2. The side of a square circumscribed about a circle is equal to the diameter.

3. How many degrees in each angle of a regular polygon of 3 sides ? of 6 sides ? of 8 sides ? of n sides ?

4. The diagonals which join alternate vertices of a regular hexagon form an equilateral triangle.



5. The side of an equilateral triangle circumscribed about a circle is twice the side of an inscribed equilateral triangle.

6. The apothem of a regular hexagon inscribed in a circle is equal to half the side of an inscribed equilateral triangle.

7. The radius of the circle circumscribed about an equilateral triangle is equal to $\frac{2}{3}$ of the altitude; and the radius of the inscribed circle is equal to $\frac{1}{3}$ of the altitude.

8. A regular polygon is symmetrical with respect to the bisector of any angle.

9. An axis of symmetry of a regular polygon has the same number of vertices on each side of it.

10. The perpendicular bisectors of the sides of a regular polygon, and the bisectors of the angles pass through the center of the polygon.

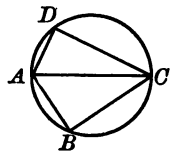
11. If a regular polygon has an even number of sides, (1) any axis of symmetry which bisects an angle is a diagonal; (2) any axis which is the perpendicular bisector of one side is the perpendicular bisector of one other side also.

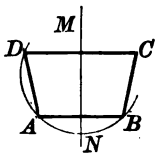
12. If a regular polygon has an odd number of sides, every axis of symmetry bisects one angle and is perpendicular to, and bisects one side.

13. If a regular polygon has an even number of sides, its center is a center of symmetry of the polygon.

14. If a regular polygon has an even number of sides, any two sides which have an equal number of vertices between their extremities are parallel.

15. If two opposite angles of a quadrilateral inscribed in a circle are equal, the diagonal which does not join their vertices is a diameter of the circle.



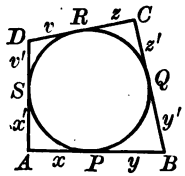


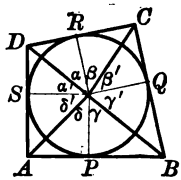
16. A circle may be circumscribed about any isosceles trapezoid.

Describe a circle through three vertices and prove by symmetry that it must pass through the other vertex.

17. The sum of two opposite sides of a quadrilateral circumscribed about a circle is equal to the sum of the other two sides.

$$x + y + v + z = x' + y' + v' + z'. \quad \text{Why?}$$





18. The angles subtended at the center of a circle by any two opposite sides of a quadrilateral circumscribed about a circle are supplementary.

$$\alpha + \beta + \gamma + \delta = \alpha' + \beta' + \gamma' + \delta'. \quad \text{Why?}$$

LOCI OF POINTS

316. A locus. The figure traced by a point which moves according to a given condition is called the *locus* of the point.

Thus if a point moves at a distance of 2 inches from a given point O, its locus is a circle with center O and radius 2 inches.

Similarly, if a point moves at a distance of 1 inch from a straight line AB, its locus is the pair of straight lines, parallel to, and at the distance of 1 inch from AB.

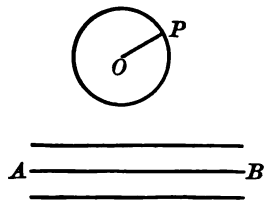


FIG. 146.

317. Test of a locus. In order that a figure shall fulfill the requirements of a locus, it is necessary and sufficient to prove that:

I. *Every point on the figure satisfies the given condition.*

II. *Conversely, every point which satisfies the given condition is on the figure.*

318. THEOREM. *The locus of a point (in a plane) at a given distance from a given point is a circle described from that point as a center, with a radius equal to the given distance.* § 18

319. THEOREM. *The locus of a point at a given distance from a given straight line is a pair of straight lines on opposite sides of the given line, parallel to it, and at the given distance from it.*

For every point on CD or $C'D'$ is at a distance a from AB. Why? Moreover, no point not on CD or $C'D'$ is at a distance a from AB. Why?

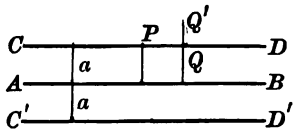


FIG. 147.

320. THEOREM. *The locus of a point equidistant from two given parallel lines is a line parallel to each of them and passing through a point midway between them.*

Show (1) that any point on EF is equidistant from AB and CD; (2) that every point equidistant from AB and CD is on EF.

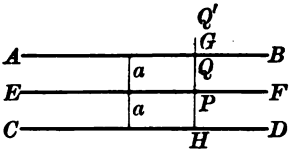


FIG. 148.

321. Use of loci. The notion of a locus is useful in locating points which satisfy two or more conditions.

EXAMPLE: To find a point 2 inches from a given point O and 1 inch from from a given straight line AB.

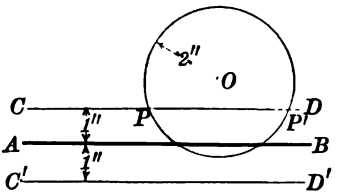


FIG. 149.

SOLUTION: Since the required point is 2 inches from O it must be on the circle described from the center O with a radius of 2 inches (§ 318); and since it is 1 inch from AB it must be on one of the lines on opposite sides of AB, parallel to it, and 1 inch from it (§ 319). Therefore, if there exists a point which satisfies both conditions, it must be where these loci meet, as P and P'. (If these loci do not meet, the solution is, of course, impossible.)

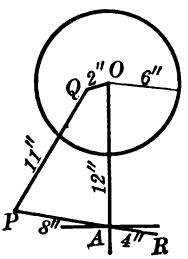
Hence: *To locate a point satisfying two or more conditions, find, if possible, the locus of a point satisfying each condition; if these loci have one or more points in common, each of these points is a solution.*

322. **EXERCISES**

1. To find a point 1 inch from a given point and equidistant from two parallel lines 2 inches apart.

2. To find a point 1 inch from a given straight line and equidistant from two parallel lines 2 inches apart.

3. Two circles have radii 1 inch and $1\frac{1}{2}$ inches, respectively, and the distance between their centers is three inches. To find a point one inch from each circle.



4. The drawing shows a design of an ordinary foot power used on a sewing machine. Find the loci of points P, Q, and R when the machine is in motion.

5. Given a circle with radius equal to 1 inch and a straight line cutting the circle. To find one or more points each at a distance of $\frac{1}{2}$ inch from the straight line and from the circle. What is the maximum number of solutions the problem can have? What is the minimum number?

323. THEOREM. *The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line segment joining them.*

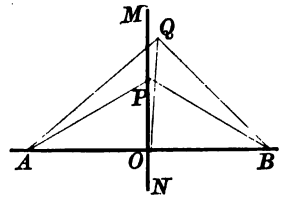


FIG. 150.

Given two points A and B, and $MN \perp AB$ at the mid-point O.

To prove that MN is the locus of a point equidistant from A and B.

Proof. 1. Let P be any point on MN. Join PA and PB.

2. Then $\triangle AOP \equiv \triangle BOP$, and $AP = BP$. Why?

3. This satisfies the first test of a locus. § 317

4. Suppose there exists a point Q not on MN which is equidistant from A and B. Join QA, QO, and QB.

5. Then $\triangle AOQ \equiv \triangle BOQ$, and $\angle QOA = \angle BOQ$. Why?

6. $\therefore QO \perp AB$. Why?

7. But $MO \perp AB$. Why?

8. $\therefore$ the supposition that there exists a point Q not on MN which is equidistant from A and B leads to a contradiction and is false. Why?

9. That is, every point which is equidistant from A and B lies on MN, which satisfies the second test of a locus. § 317

COROLLARY. *There is one and only one point which is equidistant from three given non-collinear points.*

324. THEOREM. *The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.*

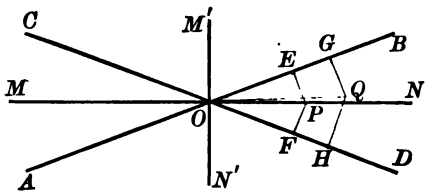


FIG. 151.

Given two straight lines AB and CD intersecting in O, and MN and $M'N'$ bisecting the angles formed by these lines.

To prove that the pair of lines MN and $M'N'$ is the locus of a point equidistant from AB and CD.

Proof. 1. Let P be any point on MN (or $M'N'$). Draw $PE \perp AB$, and $PF \perp CD$.

2. Then rt. $\triangle POE \equiv$ rt. $\triangle POF$, and $PE = PF$. Why?

3. This satisfies the first test of a locus. § 317

4. Suppose there exists a point not on MN (or $M'N'$) which is equidistant from AB and CD. Join QO and draw $QG \perp AB$, and $QH \perp CD$.

5. Then rt. $\triangle QOG \equiv QOH$, and $\angle QOG = \angle HOQ$. Why?

That is, OQ bisects one of the angles formed by AB and CD.

6. But MN and $M'N'$ bisect these angles. Why?

7. $\therefore$ the supposition that there exists a point not on MN (or $M'N'$) which is equidistant from AB and CD leads to a contradiction, and is false. Why?

8. That is, every point which is equidistant from AB and CD lies on MN or $M'N'$, which satisfies the second test of a locus. § 317

325. THEOREM. *The locus of the vertex of a right triangle opposite a given hypotenuse is a circle described on the given hypotenuse as a diameter.*

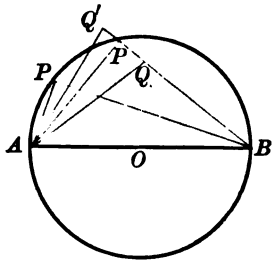


FIG. 152.

(The proof is left to the student.)

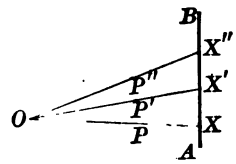
Show that any point P on the circle satisfies the requirement of the locus, and that any point Q not on the circle does not satisfy this requirement.

326. EXERCISES

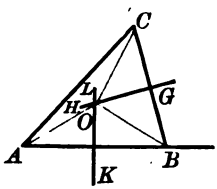
1. The perpendicular bisector of the base of an isosceles triangle passes through the opposite vertex.

2. Two points each equidistant from the extremities of a line segment determine the perpendicular bisector of the segment.

3. What is the locus of the mid-point of a segment drawn from a point to a straight line which does not pass through that point?



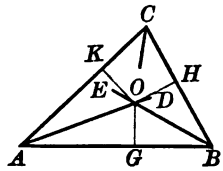
4. A line segment of constant length slides so as to have its extremities constantly resting on two lines perpendicular to each other. What is the locus of the mid-point of the segment? See Ex., § 213.

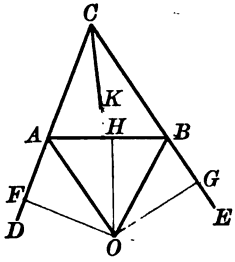


5. The perpendicular bisectors of the sides of a triangle meet in a point.

Let KL and GH be the perpendicular bisectors of AB and BC, respectively. GH and KL must meet in some point O. Why? Join OA, OB, and OC. Then $OA = OB$, and $OC = OB$. Why? $\therefore OA = OC$. Why? $\therefore O$ lies on the perpendicular bisector of AC. Why?

6. The bisectors of the angles of a triangle meet in a point. Let AD and BE be the bisectors of $\angle A$ and B, respectively. AD and BE must meet in some point O. Why? Draw $\perp$s OG, OH, and OK to AB, BC, and CA, respectively. Now proceed as in Ex. 5.



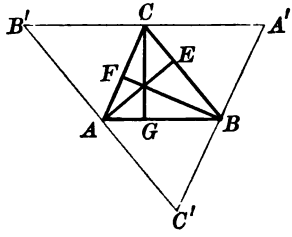


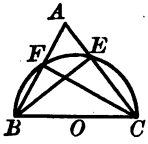
7. The bisectors of any two exterior angles of a triangle and the bisector of the opposite interior angle meet in a point.

OF = OG

Prove $\triangle$ DOC

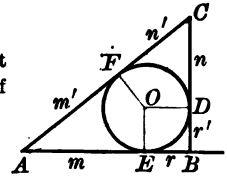
8. The altitudes of a triangle meet in a point. Let ABC be the given triangle. Through the vertices A, B, C draw lines parallel to the opposite sides forming the triangle $A'B'C'$. Then $AE \perp B'C'$, $BF \perp A'C'$, and $CG \perp A'B'$. Why? Now show that A, B, C are the mid-points of $B'C'$, $C'A'$, and $A'B'$, respectively, and apply Ex. 5.



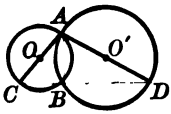


9. If a circle is described on one side of a triangle as a diameter, the circle passes through the feet of the perpendiculars drawn to the other two sides from the opposite vertices.

10. The diameter of the circle inscribed in a right triangle is equal to the difference between the sum of the legs and the hypotenuse.

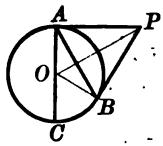


11. The median drawn to the legs of a trapezoid circumscribed about a circle is equal to one fourth the perimeter of the trapezoid (§ 249).

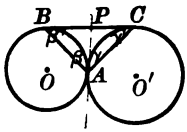


12. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the line joining the ends of the diameters passes through the other point of intersection.

Join BC and BD and prove that CBD is a straight line.

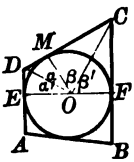


13. If two tangents are drawn to a circle from any point, the angle formed by the tangents is double the angle formed by the chord joining the points of contact, and the diameter drawn through either point of contact.



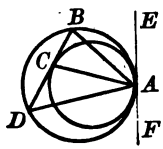
14. Two circles are tangent externally at A; B and C are points of contact of a common tangent to the two circles. Prove $\angle BAC$ is a right angle.

Draw the common tangent PA. Then $\beta = \beta'$, and $\gamma = \gamma'$. Why? But $\beta' + (\beta + \gamma) + \gamma' = 2\,R$. Why?

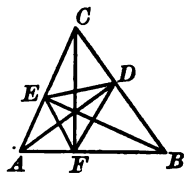


15. Each leg of a trapezoid circumscribed about a circle subtends a right angle at the center of the circle.

Draw a diameter perpendicular to AD. This diameter will pass through the points of contact E and F. Why? $\alpha' + \alpha + \beta + \beta' = 2\,R$. Why?

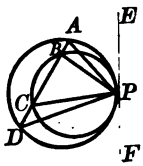


16. Two circles are tangent internally, and a chord of the greater circle is tangent to the less. Prove that the chord is divided at its point of contact into segments which subtend equal angles at the point of contact of the circles.

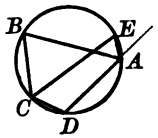


17. The perpendiculars from the vertices of a triangle to the opposite sides bisect the angles of the triangle formed by joining the feet.

Describe circles on the three sides as diameters.

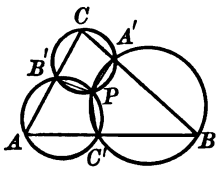


18. If two circles are tangent internally at P, and any straight line cuts the circles in the points A, B, C, D; then the angle APB is equal to angle CPD.



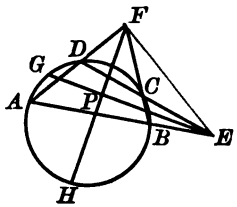
19. The bisectors of any angle of an inscribed quadrilateral and the opposite exterior angle meet on the circle.

Show that $\angle CEA$ and CDA are supplementary.



20. A', B', C' are points on the sides BC, CA, AB, of a triangle ABC, respectively. Prove that the circles passing through $AB'C'$, $BC'A'$, $CA'B'$ pass through a common point.

Two circles $AB'C'$ and $BC'A'$ must pass through a common point P. Why? Join PA', PB', and PC'. Show that $\angle B'CA'$ and $A'PB'$ are supplementary.



21. The bisectors of the angles formed by producing the opposite sides of a quadrilateral inscribed in a circle to meet, are perpendicular.

Draw FE. Then

$\angle ADC = \angle DFE + \angle FEC$, and

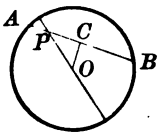
$\angle CBA = \angle BFE + \angle FEB$. Why?

$\therefore \angle ADC + \angle CBA = 2R =$

$2\angle PFB + 2\angle BFE + 2\angle FEC + 2\angle CEP$. Why?

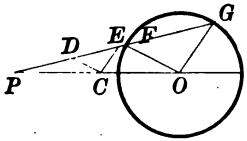
$\therefore \angle PFB + \angle BFE + \angle FEC + \angle CEP = R$.

$\therefore \angle EPF = R$. Why?



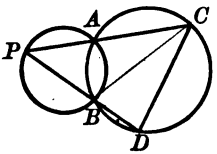
22. Find the locus of the mid-point of a chord which passes through a fixed point within a circle. Apply § 325.

23. Find the locus of the mid-point of a line segment joining a fixed point with a point on a circle.



Let O be the center of the given circle and P the fixed point. Also let PG be any segment joining P to the circle. Bisect PO, PF, and PG at C, D, and E, respectively, and draw CD, CE, OF, and OG.

Show that $CD = CE = \frac{1}{2} OG$. Hence D and E are points on the circle described from the center C with a radius equal to $\frac{1}{2} OG$.



24. Two circles intersect in the points A and B. From P any point on one circle segments PAC and PBD are drawn terminating in the other circle. Prove that the chord CD remains constant as P moves around the circle.

Show that $\angle DBC$ remains constant.

CONSTRUCTION PROBLEMS

327. Analysis of a problem. In seeking a solution of a problem, it is generally necessary to resort to an *analysis*, in order to discover known relations on which the construction depends. This consists in assuming the construction made and studying the relations between the parts of the figure, augmented by auxiliary lines when necessary. This is frequently accomplished by means of loci as illustrated in § 321.

328. PROBLEM. *To draw a tangent to a given circle from a given point outside the circle.* (Second method.)*

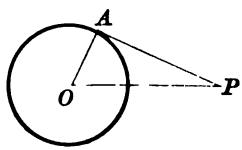


FIG. 153.

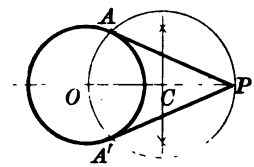


FIG. 154.

Given a point P outside the circle with center O.

Required to draw a tangent to the circle from P.

Analysis. 1. Suppose PA is the required tangent (Fig. 153).

2. Draw the radius OA, and draw OP.

3. Then $\angle OAP$ must be a right angle. Why?

4. $\therefore A$ is on the circle described on OP as a diameter. § 325.

That is, A is a point in which the given circle and the circle described on OP as a diameter intersect.

5. Hence two tangents may be drawn. Why?

Construction (Fig. 154). Let the student give it.

NOTE. The complete analysis in this solution makes a proof of the construction unnecessary.

Exercise. To construct a circle which has the radius r and which is also

a) Tangent to each of two intersecting lines AB and CD.

b) Tangent to a given straight line AB and a given circle with center O.

* For first method see § 96.

329. PROBLEM. *To circumscribe a circle about a given triangle.*

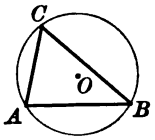


FIG. 155.

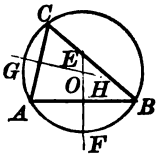


FIG. 156.

Given the triangle ABC.

Required to circumscribe a circle about ABC.

Analysis. 1. Suppose a circle with center O is circumscribed about ABC (Fig. 155).

2. Then O must be equidistant from A, B, and C. Why?

3. $\therefore$ O must be on the perpendicular bisector of AC. Why? Likewise O must be on the perpendicular bisector of AB.

4. That is, if a solution is possible, O must be the point in which the perpendicular bisectors of AB and AC meet.

(The construction and proof are left to the student.)

330. PROBLEM. *To inscribe a circle in a given triangle.*

(The solution is left to the student.)

331. PROBLEM. *To inscribe a square in a given circle.*

(The solution is left to the student.)

332. **EXERCISES**

1. To construct a right triangle, given the hypotenuse and the altitude on the hypotenuse. §§ 319, 325.

2. To construct a right triangle, given the radius of the inscribed circle and one leg.

3. To construct a right triangle, given the radius of the inscribed circle and one acute angle.

4. To construct a right triangle, given the radius of the circumscribed circle and one acute angle.

5. To construct a right triangle, given the radius of the circumscribed circle and one leg.

333. Problem. *To inscribe a regular hexagon in a given circle.*

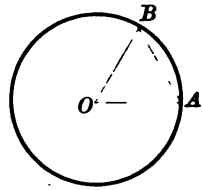


Fig. 157.

Given the circle with the center O.

Required to inscribe a regular hexagon in the circle.

Analysis. 1. Suppose AB is one side of the required regular hexagon. Draw OA and OB.

2. Then $\angle AOB = \frac{1}{6}$ of $4\ R$, or $60°$. Why?

3. But $\angle BAO = \angle OBA$. Why?

4. $\therefore$ $\measuredangle$s BAO and OBA are each equal to $60°$. Why?

5. $\therefore$ $\triangle\ OAB$ is equilateral, and $AB = OA$. Why?

That is, one side of the regular hexagon inscribed in the circle is equal to the radius.

(The construction is left to the student.)

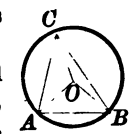
334. EXERCISES

1. To inscribe an equilateral triangle in a given circle.

2. To inscribe a regular octagon in a given circle.

3. To construct an equilateral triangle, (*a*) given the radius of the circumscribed circle; (*b*) given the radius of the inscribed circle.

4. To construct a triangle, given two angles and the radius of the circumscribed circle.



Analysis: Suppose O is the center of the circumscribed circle, and that $\measuredangle$s BAC and CBA are known. Draw OA, OB, and OC. Then $\angle BOC = 2\angle BAC$, and $\angle COA = 2\angle COB$. (Why?) Hence $\measuredangle$s BOC and COA are known.

5. To construct a triangle, given two angles and the radius of the inscribed circle.

6. To construct a circle which shall be tangent to two given parallel lines AB and CD and pass through a given point P.

335. PROBLEM. *Upon a given line segment as a chord, to describe an arc of a circle in which a given angle may be inscribed.*

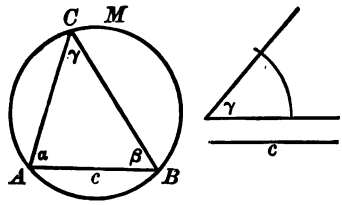


FIG. 158.

Given the line segment c and the angle γ.

Required on c as a chord, to describe an arc of a circle in which γ may be inscribed.

Analysis. 1. Suppose ACB is the required arc, that $AB = c$, and $\angle ACB = \gamma$.

2. Then $\angle ACB$ is constant and equal to γ for any position of C on the arc AMB. Why?

3. And $\alpha + \beta = 2R - \gamma$. Why?

4. Hence the circle circumscribed about any triangle ABC in which $AB = c$ and $\alpha + \beta = 2R - \gamma$ will be the required circle.

(The construction is left to the student.)

336. **EXERCISES**

1. To construct a triangle, given one side, the altitude on that side, and the opposite angle.

2. To construct a triangle, given one side, the median to that side, and the opposite angle.

3. Solve § 335 by means of § 293.

4. At a given point on a circle to draw a tangent when the center is inaccessible.

To construct a circle which has the radius r and which is also:

5. Tangent to a given straight line AB and passes through a given point P.

6. Tangent to a given circle with center O and passes through a given point P.

To construct a circle which shall be:

7. Tangent to three given lines two of which are parallel.

8. Tangent to a given line AB at P and passes through a given point Q.

9. Tangent to a given circle at P and passes through a given point Q.

10. Tangent to two given lines AB and CD and tangent to one of them at a given point P.

11. Tangent to a given straight line AB and tangent to a given circle at a given point P.

12. Tangent to a given straight line AB at P and also tangent to a given circle with center O.

13. To construct within a given circle three congruent circles, so that each shall be tangent to the other two and also the given circle.

CHAPTER VI

AREAS OF POLYGONS

MEASUREMENT

337. We have confined ourselves hitherto largely to the consideration of the properties of figures that result more or less directly from the notion of *congruence.* We now turn our attention to the problem of *comparing* geometric magnitudes that are not congruent. We seek an answer to such questions as: Which of two rectangles incloses the greater area? How much greater is the area of the larger one? How may we compare the area of a given trapezoid with that of a given triangle? etc. Such questions arose historically before the development of geometry as a science, as in the earliest periods of human history it was necessary to have means of comparing the areas of fields. Such questions, indeed, gave the motive for the development of geometry in a systematic way. The word "geometry" in the original Greek, in fact, means the "measurement of land."

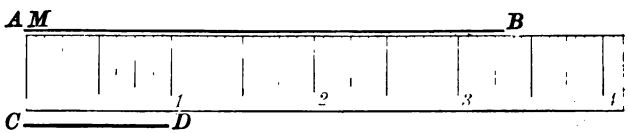


FIG. 159.

338. Ratio. Commensurable case. Let us consider first how we may compare the length of two line segments, AB and CD (Fig. 159). The natural answer is: "By measuring them, with a foot rule, for example." Suppose we find in this way that CD is just one inch long, while AB measures exactly $3\frac{5}{16}$ inches. When analyzed this process means that, if CD is divided into

16 equal parts AM, it is found that AB contains exactly 53 ($= 16 \times 3 + 5$) of these parts. By virtue of the fact that the segment AM is an aliquot part of both AB and CD, AM is called a *common measure* of AB and CD. This leads to the following definition:

If two line segments a, b are such that they have a common measure x (*i.e.* a segment which is an aliquot part of both a and b), the two segments are said to be *commensurable*. If x is contained in a exactly m times and in b exactly n times, where m and n are whole numbers, the *ratio of a to b* is the fraction $\frac{m}{n}$. This is written $\frac{a}{b} = \frac{m}{n}$. This relation is also expressed by the phrase "*a is to b as m is to n*, or $a : b = m : n$.

In the special case given above we should write $\frac{AB}{CD} = \frac{53}{16} = 3\frac{5}{16}$, and we may say AB is to CD as 53 is to 16. It should be noted further that if the fraction $\frac{m}{n}$ is not in its lowest terms, the common factor may be removed without affecting the ratio as above defined. Suppose m and n have the common factor 3 and suppose that $m = 3m'$ and $n = 3n'$. If a contains x exactly $3m'$ times, it will contain $3x$ just m' times; similarly b would contain $3x$ just n' times. That is, $3x$ would be a common measure for a and b, which would give the ratio $\frac{a}{b} = \frac{m'}{n'}$; but $\frac{m}{n} = \frac{3m'}{3n'} = \frac{m'}{n'}$. The student should make this clear to himself by considering some simple special cases.

339. Ratio. Incommensurable case. Two line segments, a and b, may not have a common measure. We shall show later in this chapter (§ 370) that the side and the diagonal of a square are two segments of this kind.

Two segments which have no common measure are said to be *incommensurable*.

The ratio of two incommensurable segments is defined in a manner which is a natural extension of the process used to define

ratio in the case of commensurable segments. It will be sufficient for our purpose to illustrate the process by a special case. Let the student construct carefully a square $ABCD$, whose side AB measures 10 cm., and draw the diagonal AC. If he measures the diagonal, he will find that its length is between 14 cm. and 15 cm. The ratio of the diagonal to the side is therefore said to be between $\frac{14}{10}$ and $\frac{15}{10}$, or $\frac{AC}{AB}$ is between 1.4 and 1.5. Now suppose AB to be divided into 100 equal parts (each equal to one millimeter); it will be found that the diagonal is longer than 141 mm., but shorter than 142 mm. The ratio $\frac{AC}{AB}$ is, therefore, a number between $\frac{141}{100}$ and $\frac{142}{100}$, *i.e.* between 1.41 and 1.42. Suppose this process could be continued indefinitely. If AB is divided into 1000 equal parts, and these parts applied to AC, beginning at A, the point C must fall on a perfectly definite one of these parts. It would be found that now we should place the ratio $\frac{AC}{AB}$ between 1.414 and 1.415; if AB is divided into 10,000 equal parts, we should find $\frac{AC}{AB}$ to lie between 1.4142 and 1.4143, and so on.

We have in this way defined the ratio $\frac{AC}{AB}$ to be a number *lying between* the successive pairs of numbers:

1.4	and 1.5
1.41	and 1.42
1.414	and 1.415
1.4142	and 1.4143

These pairs of numbers differ successively by .1, by .01, by .001, by .0001. This process then makes it possible to define the ratio $\frac{AB}{CD}$ to *as high a degree of approximation as may be desired,* since the process may be thought of as continued indefinitely. Such a process defines a definite number, which is the ratio $\frac{AC}{AB}$ sought.

340. It should be noted that the ratio of two commensurable segments is a *rational* number, *i.e.* either an integer or a fraction in which the numerator and denominator are both integers; whereas the ratio of two incommensurable segments is an *irrational* number, *i.e.* a number which is not equal to any rational number, but which may be represented approximately by a rational number to as high a degree of approximation as may be desired. Thus the ratio considered above:

$$\frac{\text{diagonal of square}}{\text{side of square}} = \sqrt{2},$$

as will appear later (§ 370).

341. Measurement of lengths. The process described in § 339 may clearly be applied to any two line segments. The word *ratio* is then defined for any two line segments. We are thus led to the following definition:

The *measure* of a line segment *a in terms of a given segment u chosen as a unit* is the ratio of the given segment a to the unit segment u.

342. Ratio of two areas. Quite similar considerations govern the notion of the ratio of two areas and the measurement of areas. We begin by comparing the area of a square with that of a rectangle. Consider the square $AHIK$ and the rectangle $ABCD$ (Fig. 160). Suppose the side AH of the square and the sides AB, AD of the rectangle have a common measure AL, which is contained exactly m times in AH and n times in AB and p times in AD. (In the figure $m=2$, $n=4$, $p=3$.) It is then apparent by drawing lines parallel to the sides as shown in the figure that the rectangle can be divided into $n \cdot p (4 \cdot 3 = 12)$ squares of side AL and that the square will contain exactly $m^2 (2 \cdot 2 = 4)$ of these smaller squares. Here the smaller square

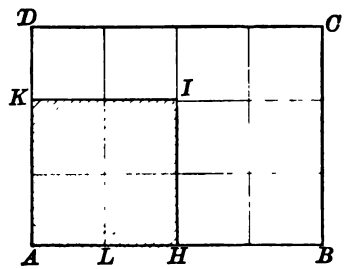


FIG. 160.

is evidently a *common measure* of the rectangle and the square and we write the *ratio*

$$\frac{\text{area of rectangle}}{\text{area of square}} = \frac{n \cdot p}{m^2} \left(= \frac{12}{4} \text{ in figure}\right).$$

Here, by virtue of the fact that a common measure exists, the rectangle and square are said to be *commensurable*. It is not necessary at this point to discuss the ratio of incommensurable areas.

343. Measurement of areas. As a unit of measure for measuring areas we use a *square*. If the unit of length is the inch, the corresponding unit of area is a square whose side measures one inch and is called the *square inch;* in a similar way we obtain as units the square foot, the square yard, the square mile, etc. To find the measure of an area in terms of a given unit area is to find the ratio of the area to the unit area. Thus in Fig. 160, if AH is 1 in., the area of the rectangle is $\frac{12}{4} = 3$ square inches.

344. Area of a rectangle. I. *Commensurable sides.* It is now readily seen that, if the sides of a rectangle are commensurable with the unit of length, *the area of the rectangle in terms of the corresponding square unit is equal to the product of the measures of two adjacent sides; i.e.* if the sides measure b units and a units, respectively, the area of the rectangle is $b \cdot a$ square units.

In the example of § 342, using AH as the unit of length, the measures of the sides are $AB = \frac{n}{m}\left(= \frac{4}{2} \text{ or } 2\right)$ and $AD = \frac{p}{m}\left(\frac{3}{2} = 1\frac{1}{2}\right)$, respectively, the segment AL being the common measure of the unit AH and the sides AB and AH. The unit square on AH, contains $m^2 (= 4)$ of the squares on AL, the rectangle contains $np (= 12)$ of these squares; the area of the rectangle is, therefore, $\frac{np}{m^2}\left(= \frac{12}{4}\right)$ square units. But $\frac{np}{m^2} = \frac{n}{m} \cdot \frac{p}{m} = \left(\frac{12}{4} = \frac{4}{2} \cdot \frac{3}{2}\right)$. This proves the statement at the beginning of this section.

II. *Incommensurable sides.* If the sides (one or both) of the rectangle are incommensurable with the unit of length, the situation is not so simple. But it can be shown that in this case also the area of the rectangle is given by the product of the two sides. Consider for example (Fig. 161) a rectangle $ABCD$ whose sides are $AD = \sqrt{5} = 2.2361 \cdots$ and $AB = \sqrt{11} = 3.3166 \cdots$.

From the figure it is clear that the area of the rectangle $ABCD$ lies between the areas of the rectangle whose sides are 2 and 3 and the rectangle whose sides are 3 and 4, *i.e.* between 6 and 12. If the unit of measure be divided into 10 equal parts, the sides of our rectangle are seen to lie between 2.2 and 2.3 and between 3.3 and 3.4, respectively, and the area sought lies between $2.2 \times 3.3 = 7.2$ and $2.3 \times 3.4 = 7.8$. The difference between these two areas is indicated in the figure by the doubly shaded strip. As we take closer and closer rational approximations to the sides of the rectangle we obtain successively pairs of rectangles *between the areas of which in every case the area of the rectangle ABCD must lie*, and, moreover, the rectangles which form these successive pairs *differ by a smaller and smaller amount, as the process proceeds.*

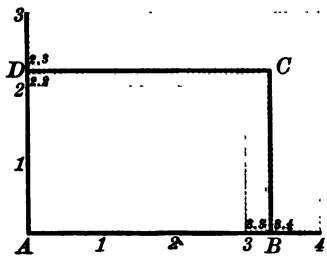


FIG. 161.

Reference to the adjoining table will show explicitly how this takes place. The first column of this table gives the fractional part of the unit used as an approximate common measure; the second column gives in each case the area of a rectangle less than that of $ABCD$, the third column, the area of a rectangle greater than that of $ABCD$. It will be seen that the pairs of corresponding areas in the last two columns agree first in the units' place, then in first decimal place, then in the second decimal place, and so on.

LINEAR MEASURE	APPROXIMATE AREA	
1.	6.	12.
0.1	7.2	7.8
0.01	7.38	7.43
0.001	7.414	7.420
0.0001	7.4162	7.4168

We conclude, therefore, that the area of the rectangle $ABCD$ is obtained by taking the product of the measures of the two sides, even when one or both of the sides are incommensurable with the unit of length. This is the meaning to be attached to the following theorem:

345. THEOREM. *The area of any rectangle is equal to the product of its base by its altitude.*

346. Equal polygons. Two polygons which have the same area are said to be *equal.*

Equal polygons are *not* necessarily congruent. Thus a rectangle whose adjacent sides are 4 and 16 is equal to a square whose side is 8, for each has the area 64 (§ 345); but obviously they are not congruent.

347. Base and altitude of a parallelogram. Since a parallelogram is a special form of trapezoid, any side of the parallelogram may be designated as the *base;* the (corresponding) *altitude* is then the perpendicular distance between the base and the opposite side (§§ 228, 229).

348. THEOREM. *The area of a parallelogram is equal to the product of its base by its altitude.*

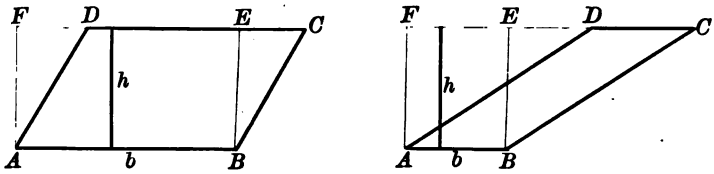


FIG. 162.

Given the parallelogram $ABCD$ with the base b and the altitude h.

To prove that the area of parallelogram $ABCD = bh$.

Proof. 1. At A and B erect perpendiculars, intersecting CD (produced if necessary) in F and E, respectively.

2. Then $ABEF$ is a rectangle with base b and altitude h. Why?

3. Since $AF = BE$, and $AD = BC$. Why?
$\therefore$ rt. $\triangle ADF \equiv$ rt. $\triangle BCE$. Why?

4. $\therefore$ parallelogram $ABCD =$ rectangle $ABEF$. Why?

5. But the area of rectangle $ABEF = bh$. § 345

6. $\therefore$ the area of parallelogram $ABCD = bh$. Why?

349. COROLLARY 1. *Parallelograms which have equal bases and altitudes are equal (not necessarily congruent).*

350. COROLLARY 2. *If two parallelograms have equal bases, the ratio of their areas is equal to the ratio of their altitudes; if they have equal altitudes, the ratio of their areas is equal to the ratio of their bases.*

351. Base of a triangle. Any side of a triangle may be called the *base* of the triangle with reference to the *altitude* drawn to that side.

352. THEOREM. *The area of a triangle is equal to half the product of its base by its altitude.*

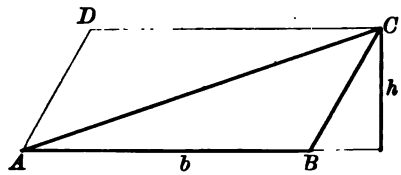


FIG. 163.

Given the triangle ABC, with base b and altitude h.

To prove that the area of $\triangle ABC = \frac{1}{2}bh$.

SUGGESTION: With AB and BC as adjacent sides construct the parallelogram $ABCD$. (The proof is left to the student.)

353. COROLLARY 1. *Triangles which have equal bases and altitudes are equal (not necessarily congruent).*

354. COROLLARY 2. *A triangle is equal to half of a parallelogram having an equal base and an equal altitude.*

355. COROLLARY 3. *If two triangles have equal bases, the ratio of their areas is equal to the ratio of their altitudes.*

356. COROLLARY 4. *If two triangles have equal altitudes, the ratio of their areas is equal to the ratio of their bases.*

357. EXERCISES

1. A median divides a triangle into two equal parts.

2. The segments which join the mid-point of either diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equal parts.

3. A straight line drawn through the mid-points of two adjacent sides of a parallelogram cuts off a triangle equal to one eighth of the given parallelogram.

358. THEOREM. *The area of a trapezoid is equal to half the product of the sum of its bases by its altitude.*

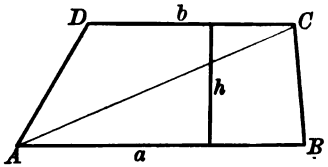


FIG. 164.

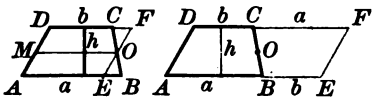
Given the trapezoid $ABCD$ with bases a, b, and altitude h.

To prove that the area $ABCD = \frac{1}{2}(a + b)h$.

SUGGESTION: Draw the diagonal AC. h is the common altitude of $\triangle ACB$ and ACD.

359. COROLLARY. *The area of a trapezoid is equal to half the product of its altitude by the median drawn to the legs.* § 249

360. EXERCISES

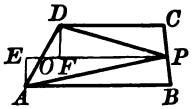


1. Prove § 358, using the adjoining figures.

2. The median drawn to the bases of a trapezoid divides the trapezoid into two equal parts.

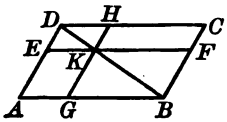
3. Every straight line drawn through the intersection of the diagonals of a parallelogram divides the parallelogram into two equal parts.

4. If any point within a parallelogram is joined to the four vertices, the sum of either pair of triangles having parallel bases is equal to half the parallelogram.



5. If two line segments are drawn from the midpoints of either leg of a trapezoid to the opposite vertices, the triangle thus formed is equal to half the trapezoid.

6. If straight lines are drawn through any point on a diagonal of a parallelogram parallel to two adjacent sides, the parallelograms formed on opposite sides of this diagonal are equal.



$\triangle EKD \equiv \triangle DKH$, and $\triangle KGB \equiv \triangle KBF$.

$\therefore \triangle EKD + \triangle KGB = \triangle DKH + \triangle KBF$.

361. THEOREM. *The area of a regular polygon is equal to half the product of its apothem by its perimeter.*

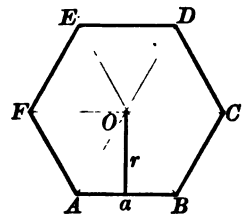


FIG. 165.

Given the regular polygon $ABCDEF$, with the apothem r and perimeter p.

To prove that the area of the polygon $ABCDEF = \frac{1}{2}rp$.

SUGGESTION: Represent the side of the polygon by a. Draw the radii OA, OB, etc. The area of each triangle thus formed is $\frac{1}{2}ra$.

362. EXERCISES

1. The area of a polygon circumscribed about a circle is equal to half the product of its perimeter by the radius of the circle.

2. The area of a rhombus is equal to half the product of its diagonals. See § 260.

3. If the mid-points of adjacent sides of a parallelogram are joined, a parallelogram is formed which is equal to one half of the given parallelogram.

4. If the base of a triangle is fixed and if the opposite vertex moves so that the area of the triangle remains constant, what is the locus of this vertex?

363. Projection. If from the end-points of a line segment perpendiculars are let fall upon a straight line, the segment cut off on the latter is called the *projection* of the line segment upon the straight line. Thus, in the figure, the segments AB' and $B'C'$ are the projections of AB and BC, respectively, upon MN.

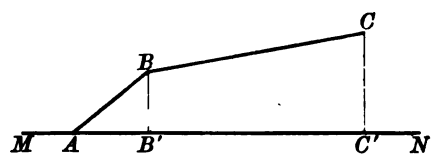


FIG. 166.

364. The area of any polygon. The area of any polygon may be found by dividing it into triangles. A common method used in practical engineering for finding the area of a polygon is as follows: Project the sides of the polygon upon some convenient base line MN (this may be a side of the polygon extended if necessary) as shown in the figure. Then the polygon $ABCDE$ is equal to the difference of the polygons $E'C'CDE$ and $E'C'CBAE$, or breaking up the polygons into trapezoids.

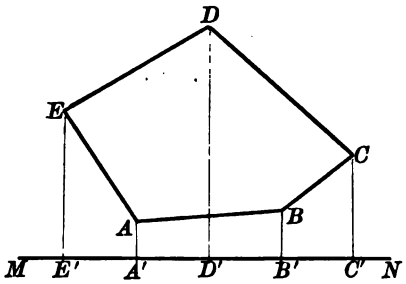
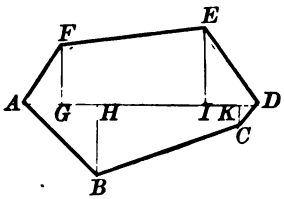


FIG. 167.

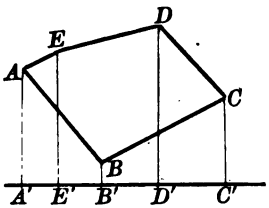
$$ABCDE = EE'D'D + DD'C'C - EE'A'A - AA'B'B - BB'C'C.$$

365. EXERCISES



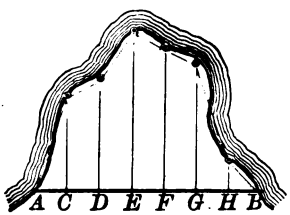
1. In the polygon $ABCDEF$, $AG = 100$ ft.; $GH = 100$ ft.; $HI = 300$ ft.; $IK = 100$ ft.; $KD = 50$ ft.; $FG = 150$ ft.; $HB = 200$ ft.; $IE = 200$ ft.; $KC = 75$ ft. Find the area.

Ans. 169,375 sq. ft.



2. Find the area of the polygon $ABCDE$, if $A'E' = 15$ yd.; $E'B' = 20$ yd.; $B'D' = 25$ yd.; $D'C' = 30$ yd.; $AA' = 50$ yd.; $EE' = 60$ yd.; $BB' = 10$ yd.; $DD' = 70$ yd.; $CC' = 40$ yd.

Ans. 2975 sq. yd.



3. A farmer sold to a power company a strip of overflow land, lying between the line AB and the river, for \$75 an acre. The surveyor measured along the line AB, driving a stake every 100 ft., at C, D, E, etc., finding AB to be 700 ft. From each of these stakes he measured the distance, at right angles to AB, to the river. These distances were found to be as follows: from C, 300 ft.; from D, 350 ft. from E, 500 ft.; from F, 450 ft.; from G, 400 ft.; from H, 100 ft. The approximate area was then computed. How much did the farmer receive for the land? *Ans.* \$361.57.

366. THEOREM. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the legs.*

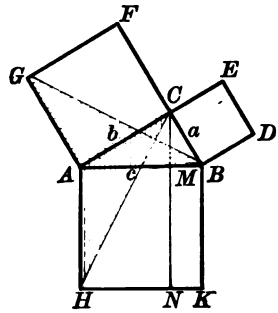


FIG. 168.

Given the rt. triangle ABC, with C the right angle.

To prove that $c^2 = a^2 + b^2$.

Proof. 1. Let $ABKH$, $ACFG$, and $BDEC$ be squares constructed on the sides c, b, a, respectively.
Draw $CN \parallel AH$ meeting AB and HK in M and N, respectively, and draw CH and BG.

2. Then in $\triangle ACH$ (vertically shaded in the fig.) and ABG (horizontally shaded in fig.)

$\angle CAH = \angle GAB$, Why?

$AC = AG$, Why?

and $AH = AB$. Why?

$\therefore \triangle ACH \equiv \triangle ABG$. Why?

3. Since FCB is a straight line parallel to GA,

sq. $ACFG = 2 \triangle ABG$. Why?

4. Also rect. $AHNM = 2 \triangle ACH$. Why?

5. $\therefore$ sq. $ACFG =$ rect. $AHNM$. Why?

In like manner it may be proved that

sq. $BDEC =$ rect. $MNKB$.

6. $\therefore$ sq. $AHKB =$ sq. $BDEC +$ sq. $ACFG$.

That is, $c^2 = a^2 + b^2$. Why?

NOTE. The above theorem, called the PYTHAGOREAN PROPOSITION, is perhaps the most famous proposition in geometry. Many proofs of the theorem

have been given, though it is probable that the general theorem was first stated and proved by *Pythagoras* (about 525 B.C.). The proof here given is usually attributed to *Euclid* (about 300 B.C.), a celebrated Greek geometer.

367. COROLLARY 1. *The square on either leg of a right triangle is equal to the difference between the square on the hypotenuse and the square on the other leg.*

368. COROLLARY 2. *The square on either leg of a right triangle is equal to the rectangle whose adjacent sides are the hypotenuse and the projection of the leg on the hypotenuse.*

For by step 5, sq. $ACFG$ = rect. $AHNM$, that is, $b^2 = cq$. In like manner $a^2 = cp$.

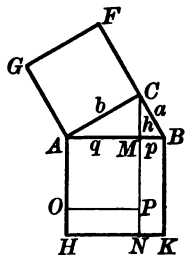


FIG. 169.

369. COROLLARY 3. *In a right triangle the square on the altitude drawn to the hypotenuse is equal to the rectangle whose adjacent sides are the projections of the legs on the hypotenuse.*

For $h^2 = b^2 - q^2$ (sq. $ACFG$ − sq. $AOPM$). § 367
But $b^2 = cq$. § 368
$\therefore h^2 = cq - q^2$ (rect. $AHNM$ − sq. $AOPM$).
$\therefore h^2 = pq$ (rect. $OHNP$).

370. COROLLARY. *The diagonal and the side of a square are incommensurable.*

For $$d^2 = s^2 + s^2 = 2\,s^2.$$
$$\therefore \frac{d}{s} = \sqrt{2}, \text{ an irrational number.}$$

That is, $\dfrac{AC}{AB}$ is an irrational number, or AC and AB are incommensurable. See § 339.

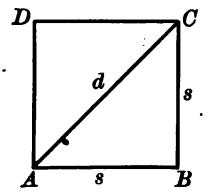
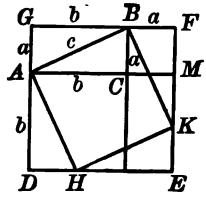


FIG. 170.

371. **EXERCISES**

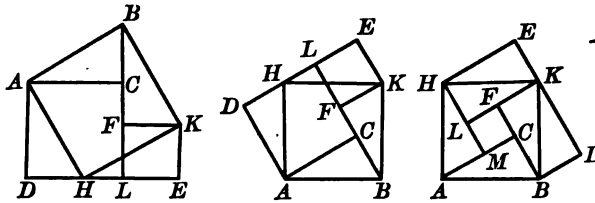
1. Prove the Pythagorean Proposition, using the adjoining figure.



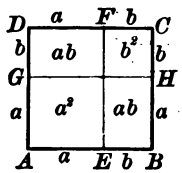
SUGGESTION: Let ACB be the given right triangle with right angle C. Construct the square $DEFG$, drawing GD and GF parallel to BC and AC, respectively. Draw AH and BK perpendicular to AB,

cutting DE and EF in H and K, respectively. Show that the right triangles shown are congruent.

2. Prove the Pythagorean Proposition, using the following figures:

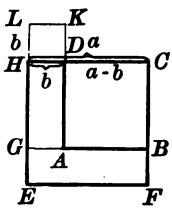


3. The square on the sum of two line segments as a side is equal to the sum of the squares on the two segments, plus twice the rectangle whose adjacent sides are the segments, respectively.



SUGGESTION: Let a and b represent the numerical measures of the two line segments. Then the square on the segment $a + b$ may be divided into rectangles and squares as shown in the figure, giving

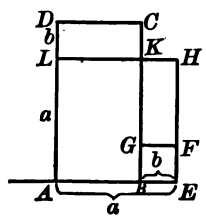
$$(a + b)^2 = a^2 + 2\,ab + b^2.$$



4. The square on the difference of two line segments as a side is equal to the sum of the squares on the line segments minus twice the rectangle whose adjacent sides are the line segments, respectively.

SUGGESTION: Sq. $ABCD$ = sq. $EFCH$ + sq. $HDKL$ − rect. $EFBG$ − rect. $GAKL$.

5. The rectangle whose adjacent sides are the sum and difference of two line segments, respectively, is equal to the difference of the squares on the two line segments.



SUGGESTION: Rect. $ABCD$ = sq. $AEHL$ + rect. $LKCD$ − rect. $GFHK$ − sq. $BEFG$.

REMARK. The three preceding exercises illustrate an important relation which exists between algebra and geometry. It will be observed that the three exercises referred to correspond exactly to three important formulæ in algebra, namely: The square of the sum of two numbers, the square of the difference of two numbers, and the product of the sum and difference of the two num-

bers. This reciprocal relation between algebra and geometry by which a single symbol, element, operation, etc., in algebra corresponds to a single symbol, element, operation, etc., in geometry is of great importance, and is made use of extensively in geometry to avoid otherwise cumbersome methods.

6. Show by a figure that

(a) $a(b + c) = ab + ac.$

(b) $(a + b)^2 - (a - b)^2 = 4\,ab.$

(c) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$

7. The square on a line segment is equal to four times the square on half the segment.

8. The square on a diagonal of a given square is equal to twice the given square.

9. If from the mid-point of any side of a triangle, straight lines are drawn parallel to the other two sides, the parallelogram thus formed is equal to one half of the triangle.

372. NUMERICAL EXERCISES

1. The base of an isosceles triangle is 10 in. and the legs are each 13 in. Find the altitude and the area. *Ans.* 12 in.; 60 sq. in.

2. The side of an equilateral triangle is 8 in. Find the area. *Ans.* 27.71 sq. in.

3. The diagonals of a rhombus are 12 in. and 16 in. Find the perimeter and the area. *Ans.* 40 in.; 96 sq. in.

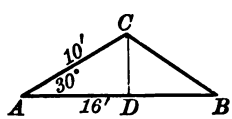
4. The bases of an isosceles trapezoid are 10 in. and 16 in., and the legs are each 5 in. Find the area. *Ans.* 52 sq. in.

5. The hypotenuse of a right isosceles triangle is 20 in. Find the area. *Ans.* 100 sq. in.

6. The hypotenuse of a right triangle is 10 ft., and one acute angle is 60°. Find the area. *Ans.* 21.65 sq. ft.

SUGGESTION. If one angle of a right triangle is 30°, the hypotenuse is double the shorter leg. Ex. 10, § 265.

7. Two sides of a triangle are 10 ft. and 16 ft., and the included angle is 30°. Find the area. *Ans.* 40 sq. ft.



SUGGESTION. Find the altitude CD.

8. Two sides of a parallelogram are 25 ft. and 40 ft., and the included angle is 120°. Find the diagonals and the area.

Ans. Area 866.0 sq. ft.; diagonals 56.79 ft., 35.00 ft.

9. A chord 30 in. long is 8 in. from the center of a circle. Find the radius. *Ans.* 17 in.

10. The radius of a circle is 25 in. How far from the center is a chord 48 in. long? *Ans.* 7 in.

11. If a chord 8 in. long is 3 in. from the center of a circle, find the length of a chord 4 in. from the center. *Ans.* 6 in.

12. Find the side and the area of a square inscribed in a circle whose radius is 10 in. *Ans.* 7.071 in.; 50 sq. in.

13. Find the area of a regular hexagon inscribed in a circle whose radius is 10 in. See § 314. *Ans.* 259.8 sq. in.

14. Find the side of an equilateral triangle inscribed in a circle whose radius is 10 in. *Ans.* 17.32 in.

15. Find the side of an equilateral triangle circumscribed about a circle whose radius is 10 in. *Ans.* 34.64 in.

16. The centers of two circles are 17 in. apart. Their radii are 2 in. and 10 in., respectively. Find the length of their common exterior tangents. See § 104. *Ans.* 15 in.

17. The centers of two circles are 25 in. apart. Their radii are 3 in. and 4 in., respectively. Find the length of their common interior tangents. See § 106. *Ans.* 24 in.

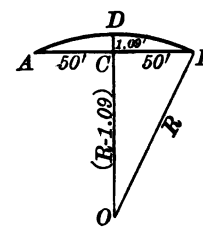
18. If, in Ex. 5, § 17, the dimensions of the room are 16 ft. wide, 20 ft. long, and 10 ft. high, find the length of the spider's path. *Ans.* 32.8 ft.

19. A ladder 25 ft. long just reaches the top of a wall, when the foot of the ladder is 7 ft. from the wall. How high is the wall? *Ans.* 24 ft.

20. A guy wire extends from a point on a telephone pole 30 feet from the base to a point on a tree 35 feet distant and 18 feet from the base of the tree. Find the length of the wire connecting the pole with the tree. *Ans.* 37 feet.

21. A ship sails east 40 miles, then north 11 miles, then east 20 miles. How far is it from the starting point? *Ans.* 61 miles.

22. In order to find the radius of a railway curve (an arc of a circle) a surveyor made the following measurements: From C, the mid-point of a 100-foot tape AB, stretched between two points on the curve, the distance CD to the arc was measured and found to be 1.09 feet. Compute the radius. *Ans.* 1147 feet.



SUGGESTION: Represent the radius by R. Then in the rt. $\triangle$ OBC, $OB = R$, $OC = R - 1.09$, $CB = 50$. Apply § 366.

CONSTRUCTION PROBLEMS

373. PROBLEM. *To construct a triangle equal to a given triangle, and having one side equal to a given line segment.*

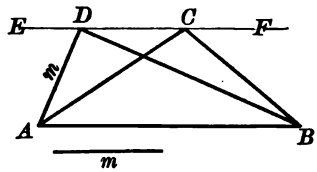


FIG. 171.

Given the triangle ABC, and the line segment m.

Required to construct a triangle equal to the triangle ABC, and having one side equal to m.

Construction. Through the vertex C (Fig. 171) draw $EF \parallel AB$. With A as a center, and a radius equal to m describe an arc cutting EF (if possible) in D. Join D to A and B.

Then $\triangle ABD$ is the triangle required.

Proof. 1. $\triangle ABD = \triangle ABC$. Why?

2. Also $BD = m$. Why?

3. $\therefore \triangle DBC$ satisfies the required conditions. Why?

Discussion. If the altitude of $\triangle ABC$ on the base AB is greater than m, the above construction will fail. Why? In this case, proceed as follows: Through the vertex C (Fig. 172) draw $EF \parallel AB$. Construct $\triangle ABD$, such that the altitude on the base AD is less than m. Then construct $\triangle AED = \triangle ABD$, making $DE = m$, as described above. $\triangle AED$ is the triangle required. Why?

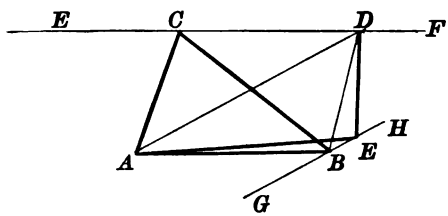


FIG. 172.

374. PROBLEM. *To construct a triangle equal to a given polygon.*

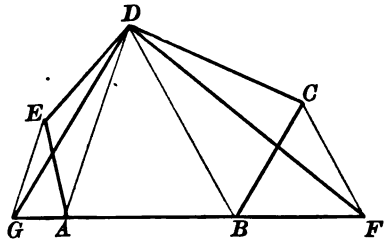


FIG. 173.

Given the polygon $ABCDE$ ....

Required to construct a triangle equal to the polygon $ABCDE$.

Construction. Suppose the polygon has five vertices. Let B, C, D be three consecutive vertices.

Draw BD, and through C draw a parallel to DB, cutting AB produced in F. Draw DF.

In a similar manner draw DG.

Then $\triangle GFD$ is the triangle required.

Proof. 1. $\triangle DBF = \triangle DBC$, and

$\triangle DAG = \triangle DAE$. Why?

2. $\therefore \triangle DBF + \triangle DAB + \triangle DAG = \triangle DBC + \triangle DAB + \triangle DAE$. Why?

That is, $\triangle GFD =$ polygon $ABCDE$.

Let the student show how to extend this method so as to apply to a polygon with six, seven, etc., vertices.

375. EXERCISES

1. To construct a right triangle equal to a given triangle.

2. To construct a triangle equal to a given triangle, and having one angle equal to a given angle.

3. To construct a right triangle equal to a given triangle, and having one leg equal to a given line segment.

SUGGESTION: First construct a triangle equal to the given triangle, and having one side equal to the given segment, § 373; then apply Ex. 1.

4. To construct an isosceles triangle equal to a given triangle and having the base equal to a given line segment.

376. PROBLEM. *To construct a square equal to a given rectangle.*

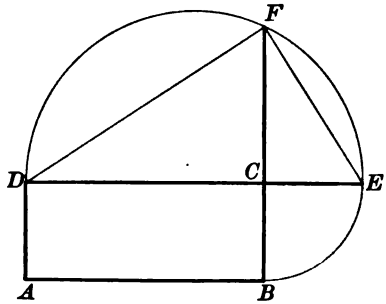


FIG. 174.

Given the rectangle $ABCD$.

Required to construct a square equal to rect. $ABCD$.

Construction. Produce DC to E, making $CE = CB$.

On DE as a diameter describe a semicircle, cutting BC produced in F.

Then CF is a side of the required square.

Proof. 1. Draw FD and FE.

2. $\angle DFE$ is a right angle and $FC \perp DE$. Why?

3. $\therefore$ the square on FC is equal to the rectangle whose adjacent sides are DC and CE, or CB. § 369.

377. PROBLEM. *To construct a square equal to the sum, or difference, of two given squares.*

The solution of this problem is left to the student.

378. EXERCISES

1. To construct a rectangle equal to a given parallelogram.

2. To construct a parallelogram equal to a given parallelogram, and having one side equal to a given line segment.

SUGGESTION: The construction is similar to that of § 373.

3. To construct a rectangle equal to a given parallelogram, and having one side equal to a given line segment.

4. To construct a parallelogram equal to a given parallelogram, and having an angle equal to a given angle.

5. To construct a parallelogram equal to a given triangle.

6. To construct a square equal to a given triangle.

SUGGESTION: First construct a parallelogram equal to the given triangle; then apply § 376.

7. To construct a square equal to a given polygon.

8. To construct a square equal to the sum of three or more given squares.

9. To construct a square equal to the sum of two or more given triangles.

10. To construct a right triangle equal to a given triangle, and having the hypotenuse equal to a given line segment. See § 325.

11. To construct a triangle equal to a given triangle, having one side equal to a given line segment, and the angle opposite that side equal to a given angle. See § 335.

12. To find a point within a triangle, such that the segments joining this point to the vertices shall divide the triangle into three equal parts.

13. To divide a given triangle into two equal parts by a line through a given point on one of the sides.

14. An attempt is made to row a boat at the rate of 4 miles an hour directly across a stream flowing at the rate of 3 miles an hour. If the stream is a mile wide, how long will it take to row across, and how far will the boat move? See Ex. 11, § 281. *Ans.* 15 minutes; $1\frac{1}{4}$ miles.

15. A sailor climbs a mast at the rate of 5 feet a second. The ship is sailing at the rate of 12 feet a second. Over what space does he actually move during 15 seconds? *Ans.* 195 feet.

16. A railway train moves southwestward at the rate of 40 miles an hour. (*a*) How fast is it moving westward? (*b*) How fast is it moving southward? *Ans.* 28.28 miles an hour.

17. A man pushes at one corner of the rear end of a street car with a force of 70 pounds, and in a direction that makes an angle of 30° with the car's line of motion. How many pounds effective force does he exert in the direction of the line of motion of the car? *Ans.* 60.6 lb.

CHAPTER VII

PROPORTION. SIMILAR POLYGONS

379. In the preceding chapters we have studied the relations between figures which have the same size and shape (congruent figures), and between figures that have the same size but which differ in shape (equal figures). We wish now to investigate the properties of figures that have the same shape, but which differ in size (similar figures).

FIG. 175.

(EUCLID)

FIG. 176.

Figures 175 and 176 are reproductions of the same portrait. Evidently they have the same shape, though they differ in size. Observe that each of the equal squares shown in Fig. 176 corresponds to one of the squares shown in Fig. 175. By actual measurement it will be found that the sides of the former are two-thirds the sides of the latter, while their areas are to each other as 4 to 9.

From this example, it will be seen that the study of figures which have the same shape, but which differ in size, is intimately connected with ratio and proportion.

380. Terms of a ratio. In the ratio $\frac{a}{b}$, a is called the *antecedent* and b is called the *consequent;* each is called a *term* of the ratio.

381. THEOREM. *In a sequence of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

For, let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=r$, where $a, b, c \cdots$ are positive numbers.

Then $a=br,\ c=dr,\ e=fr,$ and $g=hr.$ Why?

$\therefore\ a+c+e+g=br+dr+fr+hr=r(b+d+f+h).$ Why?

$\therefore\ \frac{a+c+e+g}{b+d+f+h}=r=\frac{a}{b}=\frac{c}{d}$, etc. Why?

382. A proportion. An equality of two ratios is called a *proportion.*

The proportion $\frac{a}{b}=\frac{c}{d}$, also written $a:b=c:d$, or $a:b::c:d$, is read a is to b as c is to d; a and b are also said to be *in proportion* to c and d.

383. Terms of a proportion. In the proportion $\frac{a}{b}=\frac{c}{d}$, each of the numbers a, b, c, and d is called a *term* of the proportion. The first and last terms, a and d, are called the *extremes ;* and the second and third terms, b and c, are called the *means.*

384. Fourth proportional, mean proportional, third proportional. The last term of a proportion is called a *fourth proportional* to the other three terms.

Thus, in the proportion $\frac{a}{b}=\frac{c}{d}$, d is a fourth proportional to a, b, and c.

If three numbers a, b, and c are in the proportion $\frac{a}{b}=\frac{b}{c}$, then b is called the *mean proportional* between a and c; and c is called a *third proportional* to a and b.

PROPERTIES OF A PROPORTION

385. THEOREM. *If four numbers a, b, c, and d are in proportion, as* $\frac{a}{b} = \frac{c}{d}$, *then:*

The product of the means is equal to the product of the extremes, that is

(1) $$ad = bc.$$

(Multiplying both members of the given equation by bd.)

The numbers are in proportion by alternation, that is

(2) $$\frac{a}{c} = \frac{b}{d},$$

(Dividing both members of (1) by cd.)

The numbers are in proportion by inversion, that is

(3) $$\frac{b}{a} = \frac{d}{c},$$

(Dividing both members of (1) by ac.)

The numbers are in proportion by composition; that is,

(4) $$\frac{a+b}{b} = \frac{c+d}{d}, \text{ and } \frac{a+b}{a} = \frac{c+d}{c}.$$

(Adding 1 to both members of the given equation and to equation (3), respectively.)

The numbers are in proportion by division; that is,

(5) $$\frac{a-b}{b} = \frac{c-d}{d}, \text{ and } \frac{a-b}{a} = \frac{c-d}{c}.$$

(Subtracting 1 from both members of the given equation and from equation (3), respectively.)

The numbers are in proportion by composition and division; that is,

(6) $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

(Dividing the members of equation 4 by the corresponding members of equation 5.)

REMARK. By means of the above theorem the form of a proportion may be changed in a variety of ways without altering the relation connecting the terms. Such an operation is called *transforming* the proportion by alternation, inversion, composition, etc.

EXAMPLE: Transform the proportion $\frac{2}{3} = \frac{8}{9}$ by alternation, inversion, composition, division, composition and division.

386. THEOREM. *The mean proportional between two numbers is equal to the square root of their product.*

For, let $\frac{a}{b} = \frac{b}{c}$, then $b^2 = ac$. Why? $\therefore b = \sqrt{ac}$.

387. EXERCISES

1. Find the value of x in the following proportions:

$$(a)\ \frac{3}{5} = \frac{6}{x};\ (b)\ \frac{2}{x} = \frac{x}{8};\ (c)\ \frac{b}{x} = \frac{c}{d};\ (d)\ \frac{a+b}{a-b} = \frac{x}{c}.$$

2. Are the following proportions correct? $(a)\ \frac{3}{2\frac{1}{2}} = \frac{4}{3\frac{1}{3}}$; $(b)\ \frac{2\frac{1}{2}}{3\frac{1}{3}} = \frac{4\frac{1}{2}}{5\frac{1}{3}}$? What is the test of a proportion?

3. Form proportions from the following equal products and test by the method developed in Ex. 2:

$$(a)\ my = nx;\ (b)\ 2\,ab = c^2;\ (c)\ x = \frac{2\,ab}{c};\ (d)\ x^2 = \frac{3\,a^2}{2}.$$

4. Find a fourth proportional to 2, 3, and 7; to $2\,a$, $a + c$, and c.

5. Find the mean proportional between 3 and 48; between $5\,a^2$ and $45\,b^2$.

6. Find a third proportional to 5 and 8; to $a + b$ and $a - b$.

7. Transform the following proportions by alternation, inversion, composition, division, composition and division, and test the results:

$$(a)\ \frac{5}{3} = \frac{20}{12};\ (b)\ \frac{2\,a}{b} = \frac{c}{3\,d}.$$

8. In the proportion $\frac{a}{b} = \frac{a}{c}$, prove that $b = c$.

9. Illustrate the truth of § 381 by using the equal ratios

$$\tfrac{2}{5} = \tfrac{4}{10} = \tfrac{8}{20} = \tfrac{6}{15} = \tfrac{10}{25}.$$

10. If $\frac{a}{b} = \frac{c}{d}$, prove that

$$(a)\ \frac{ma}{nb} = \frac{mc}{nd};\ (b)\ \frac{2a+b}{b} = \frac{2\,c+d}{d};\ (c)\ \frac{a}{c} = \frac{a+2\,b}{c+2\,d}.$$

388. Proportional line segments. Two line segments are said to be *divided proportionally* if the segments of the one have the same ratio as the segments of the other.

389. THEOREM. *If a straight line is drawn through two sides of a triangle parallel to the third side, it divides these sides proportionally.*

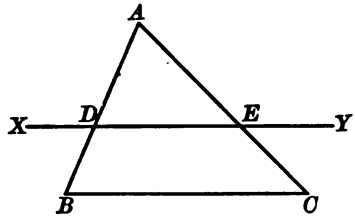


FIG. 177.

Given the $\triangle ABC$ with $XY \parallel BC$, cutting AB and AC in D and E, respectively.

To prove that $\frac{AD}{DB} = \frac{AE}{EC}$.

Proof. 1. If possible find a common measure of AD and DB. Suppose AD and DB contain this measure m times and n times, respectively.

2. Then $\frac{AD}{DB} = \frac{m}{n}$. Def. § 338

3. Imagine lines drawn through the points of division on AD and DB parallel to BC, cutting AC.

4. These lines divide AE and EC into m and n equal segments, respectively. § 251

5. $\therefore \frac{AE}{EC} = \frac{m}{n}$. Def. § 338

6. $\therefore \frac{AD}{DB} = \frac{AE}{EC}$. Why?

390. Proof of § 389 (continued). If AD and DB in the preceding figure have no common measure, that is, if AD and BD are incommensurable, the proof in the preceding paragraph is not valid. In this case we reason as follows:

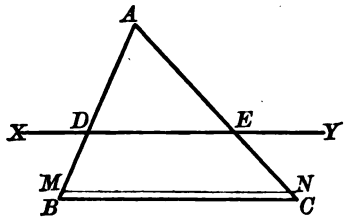


FIG. 178.

Suppose AD divided into m equal parts, and one of these parts is applied to DB n times with a remainder MB, less than one of these parts. Draw $MN \parallel BC$, cutting AC in N. Then by the preceding proof

$$\frac{DM}{AD}=\frac{EN}{AE}=\frac{n}{m}.$$

Then $$\frac{n}{m}<\frac{DB}{AD}<\frac{n+1}{m}$$

and $$\frac{n}{m}<\frac{EC}{AE}<\frac{n+1}{m}.$$

That is, $\frac{DB}{AD}$ and $\frac{EC}{AE}$ lie between $\frac{n}{m}$ and $\frac{n+1}{m}$.

$$\therefore \frac{DB}{AD}-\frac{EC}{AE}<\frac{n+1}{m}-\frac{n}{m}, \text{ or } \frac{1}{m}.$$

But, by increasing m, $\frac{1}{m}$ can be made less than any assignable positive number, however small. Consequently $\frac{DB}{AD}-\frac{EC}{AE}$ can be made less than any such number. Therefore, to suppose that $\frac{DB}{AD}$ and $\frac{EC}{AE}$ are different, leads to an absurdity. Therefore,

$$\frac{DB}{AD}=\frac{EC}{AE}, \text{ or } \frac{AD}{DB}=\frac{AE}{EC}.$$

391. The Theorem of § 389 may be extended to include the proportions obtained by transforming the proportion $\frac{AD}{DB}=\frac{AE}{EC}$ by inversion, by alternation, and by composition. Let the student

give the resulting proportions. He should note, in particular, that composition gives $\frac{AB}{AD}=\frac{AC}{AE}$, or $\frac{AB}{DB}=\frac{AC}{EC}$.

392. COROLLARY. *Three or more parallel lines in a plane cut off proportional segments on any two transversals.*

Draw $AP \parallel BH$.

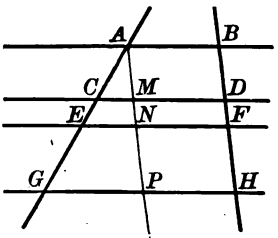


FIG. 179.

1. Then $\frac{AC}{AM}=\frac{CE}{MN}$. Why?

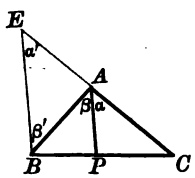
2. But $BD=AM$, and $DF=MN$. Why?

3. $\therefore \frac{AC}{BD}=\frac{CE}{DF}$. Why?

In like manner it can be proved that $\frac{CE}{DF}=\frac{EG}{FH}$. 4. $\therefore \frac{AC}{BD}=\frac{CE}{DF}=\frac{EG}{FH}$, etc. Why?

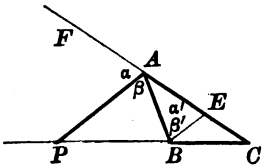
393. External and internal division of a line segment. A point P on the line segment AB is said to divide AB *internally* into the segments PA and PB; and a point P' on AB produced is said to divide AB *externally* into segments $P'A$ and $P'B$.

394. **EXERCISES**



1. The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.

SUGGESTION: It is required to prove that $\frac{PB}{PC}=\frac{AB}{AC}$. Draw $BE \parallel AP$, meeting CA produced in E. Then $\frac{PB}{PC}=\frac{AE}{AC}$, § 389. Now show that $AB=AE$.



2. The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the adjacent sides.

SUGGESTION: It is required to prove that $\frac{PB}{PC}=\frac{AB}{AC}$. Draw $BE \parallel AP$, meeting CA in E. Then $\frac{PB}{PC}=\frac{AE}{AC}$. Why? Now show that $AB=AE$.

395. THEOREM. *If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*

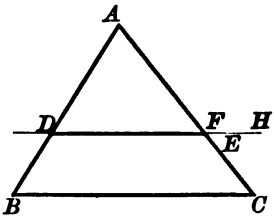


FIG. 180.

Given the $\triangle ABC$ with DE drawn so that $\frac{AD}{AB}=\frac{AE}{AC}$.

To prove that $DE \parallel BC$.

Proof. 1. Suppose DH drawn parallel to BC cutting AC in F.

2. Then $\frac{AD}{AB}=\frac{AF}{AC}$. Why?

3. But $\frac{AD}{AB}=\frac{AE}{AC}$. Why?

4. $\therefore \frac{AF}{AC}=\frac{AE}{AC}$. Why?

5. $\therefore AF=AE$. Why?
That is, E and F coincide.

6. $\therefore DE$ coincides with DF. Why?
That is, $DE \parallel BC$.

396. EXERCISES

1. The line segment joining the mid-points of two sides of a triangle is parallel to the third side.

2. The straight line which is parallel to one side of a triangle, and bisects one of the remaining sides, bisects the third side also. Compare with § 252.

3. If equal segments are laid off on the legs of an isosceles triangle, measured from the vertex, the line joining the extremities of these segments is parallel to the base.

4. From any point P within a triangle ABC, PA, PB, PC are drawn and are bisected, respectively, at A', B', C'. Prove that

$$\angle ABC = \angle A'B'C'.$$

CONSTRUCTION PROBLEMS

397. PROBLEM. *To construct a fourth proportional to three given line segments.*

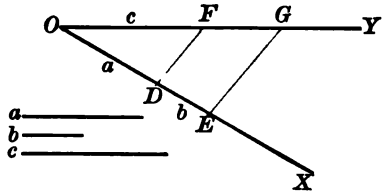


FIG. 181.

Given the line segments a, b, and c.

Required to construct a fourth proportional to a, b, and c.

Construction. Draw any convenient angle XOY.
On OX, lay off $OD = a$, and $DE = b$. On OY, lay off $OF = c$.
Draw DF and draw $EG \parallel DF$, cutting OY in G.
Then FG is the fourth proportional required.

Proof. 1. $\frac{OD}{DE} = \frac{OF}{FG}$. § 389

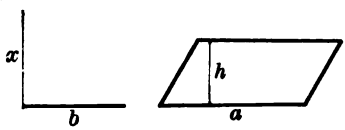
2. But $OD = a$, $DE = b$, and $OF = c$. Why?

3. $\therefore \frac{a}{b} = \frac{c}{FG}$. Why?

4. $\therefore FG$ is a fourth proportional to a, b, and c. Def.

398. EXERCISES

1. To construct a rectangle equal to a given parallelogram and having one side equal to a given line segment.



ANALYSIS. Represent the unknown side of the required rectangle by x. Then, since the areas of the two parallelograms must be the same, $bx = ah$. $\therefore \frac{b}{a} = \frac{h}{x}$. Hence x is the fourth proportional to b, a, and h, in the order named. Compare with EX. 2, § 378.

2. To construct a rectangle equal to twice a given parallelogram and having one side equal to a given line segment.

399. Problem. *To construct the mean proportional between two given line segments.*

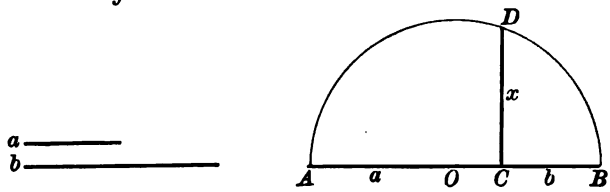


Fig. 182.

Given two line segments a, b.

Required to construct the mean proportional between a and b.

Analysis. Let x represent the required segment. Then $\frac{a}{x}=\frac{x}{b}$, by def. $\therefore x^2=ab$. Why? Hence x is the side of a square equal to the rectangle whose adjacent sides are a and b. See § 376.

(Let the student give the construction.)

Remark. It should be noted that a, b, and x are here regarded as the *measures* of the segments which they represent.

400. Problem. *To construct a third proportional to two given line segments.*

Suggestion: Construct x in the proportion $\frac{a}{b}=\frac{b}{x}$, § 397.

401. **EXERCISES**

1. To construct a square equal to $\frac{2}{3}$ of a given square.

Analysis. Let a and x represent the measures of sides of the given square and the unknown square, respectively. Then $x^2=\frac{2}{3}a^2$. Why? $\therefore \frac{a}{x}=\frac{x}{\frac{2}{3}a}$. Why? Hence, construct the mean proportional between a and $\frac{2}{3}a$.

2. To construct a square equal to $\frac{3}{4}$ of a given quadrilateral.

3. To construct a rectangle equal to $\frac{1}{2}$ of a given square, and having one side equal to a given line segment.

402. PROBLEM. *To divide a given line segment into parts proportional to any number of given segments.*

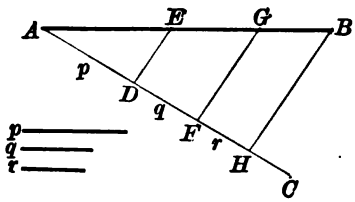


FIG. 183.

Given the line segments AB, p, q, r.

Required to divide AB into parts proportional to p, q, and r.

SUGGESTION: It is required to find points E and G on AB, such that $\frac{AE}{p} = \frac{EG}{q} = \frac{GB}{r}$.

403. EXERCISES

1. Divide a line 6 inches long into parts proportional to the numbers 2, 3, 5.

2. The sum of two line segments is equal to a given segment a, and their ratio is $\frac{m}{n}$. Construct the segments.

3. The difference of two line segments is equal to a segment d, and their ratio is $\frac{m}{n}$. Construct the segments.

4. Given segments a, b, and c. To construct a segment x, such that

$$(a)\ x = \frac{ab}{c}; \quad (b)\ x = \frac{a^2}{c}.$$

5. Given a segment a. To construct segments equal to $a\sqrt{2}$, $a\sqrt{3}$, $a\sqrt{5}$, etc.

ANALYSIS. Let x represent the required segment. Then (in the first case) $x = a\sqrt{2}$. $\therefore x^2 = 2a^2$. Why? $\therefore \frac{2a}{x} = \frac{x}{a}$. Why? Hence x is a mean proportional between a and $2a$.

6. Given two line segments a, b $(a > b)$. Construct a segment equal to $\frac{1}{2}\sqrt{ab} + 2\sqrt{a^2 - b^2}$.

SUGGESTION: Let $x = \frac{1}{2}\sqrt{ab}$, and $y = 2\sqrt{a^2 - b^2}$.

SIMILAR POLYGONS

404. Similar polygons. Two polygons of the same number of sides are said to be *similar* ($\sim$), if the angles of the one are equal, respectively, to the angles of the other, and the homologous sides are proportional. The respectively equal angles are called *homologous angles*, and the sides adjacent to these angles are called *homologous sides*.

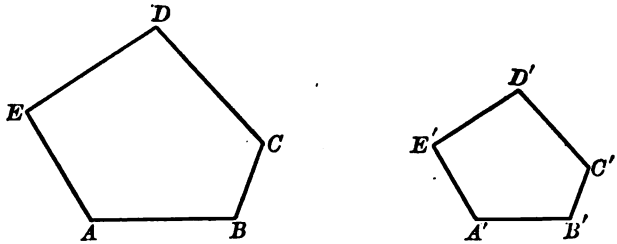


FIG 184.

The polygons $ABCDE$ and $A'B'C'D'E'$ are similar if the following conditions are satisfied:

(1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.;

(2) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$, etc.

405. Ratio of similitude. The ratio of two homologous sides in similar polygons is called the *ratio of similitude* of the polygons.

406. **EXERCISES**

1. The sides of a triangle are 4, 5, 6, respectively. In a similar triangle the side homologous to 4 is 20. Find the other two sides; also the ratio of similitude.

2. Two adjacent sides of a rectangle are 6 and 8. Find the sides of a similar rectangle whose perimeter is 42.

3. In a map drawn to the scale 1 : 100,000 what lengths will represent the sides of a county that is a rectangle 30 miles long and 15 miles wide? Give the answer to the nearest hundredth of an inch.

4. The perimeter of a rectangle is 64 inches and the ratio of the two sides is $\frac{7}{9}$. Find the sides.

407. THEOREM. *Two triangles are similar if they have two angles of the one equal, respectively, to two angles of the other.*

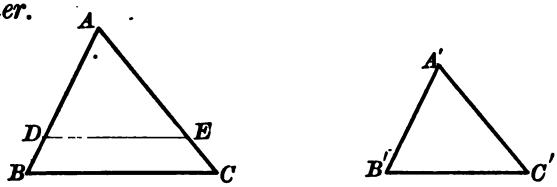


FIG. 185.

Given the $\triangle ABC$ and $A'B'C'$ with $\angle A = \angle A'$, and $\angle B = \angle B'$.

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof. 1. Imagine $\triangle A'B'C'$ placed in the position ADE, $\angle A'$ coinciding with its equal $\angle A$, B' falling at D on AB, and C' falling at E on AC.

(Let the student explain in detail how this may be done.)

2. Since $\angle EDA = \angle B$ — Why?
$DE \parallel BC$. — Why?

3. $\therefore \dfrac{AB}{AD} = \dfrac{AC}{AE}$. That is, $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$. — Why?

4. In like manner it can be proved that
$\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$.

5. $\therefore \dfrac{AB}{A'B'} = \dfrac{AC}{A'C'} = \dfrac{BC}{B'C'}$. — Why?

6. Also, since $\angle A = \angle A'$, and $\angle B = \angle B'$ — Why?
$\angle C = \angle C'$. — Why?

7. $\therefore \triangle ABC \sim \triangle A'B'C'$. — § 404

408. COROLLARY 1. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

409. COROLLARY 2. *Two isosceles triangles are similar if the vertical angle of the one is equal to the vertical angle of the other.*

410. THEOREM. *Two triangles are similar, if they have an angle of the one equal to an angle of the other and the including sides are proportional.*

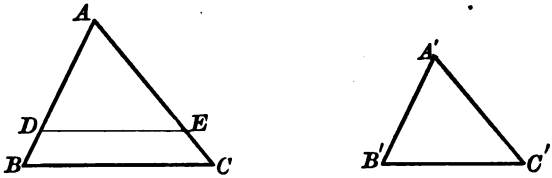


FIG. 186.

Given the $\triangle$s ABC and $A'B'C'$, with $\angle A = \angle A'$, and $\frac{AB}{A'B'} = \frac{AC}{A'C'}$.

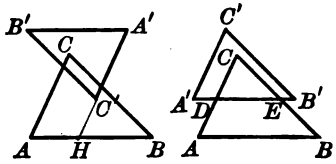
To prove that $\triangle ABC \sim \triangle A'B'C'$.

SUGGESTION: Imagine $\triangle A'B'C'$ placed in the position ADE, etc. Then, since $\frac{AB}{AD} = \frac{AC}{AE}$, $DE \parallel BC$. Why?

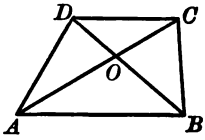
411. EXERCISES

1. Two triangles are similar if their sides are respectively parallel.

SUGGESTION: Show that two angles of the one triangle are equal, respectively, to two angles of the other.

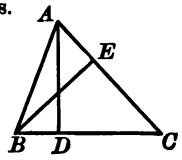


2. The diagonals of a trapezoid divide each other into proportional segments.



Show that $\triangle AOB \sim \triangle COD$. Then write the proportion resulting from the definition of similar polygons, § 404.

3. Two altitudes of a triangle are inversely proportional to the corresponding bases.



It is required to prove that $\frac{AD}{BE} = \frac{AC}{BC}$.

Show that rt. $\triangle ADC \sim$ rt. $\triangle BEC$. (This may also be proved algebraically by means of the area of the triangle.)

412. THEOREM. *Two triangles are similar, if their sides are respectively proportional.*

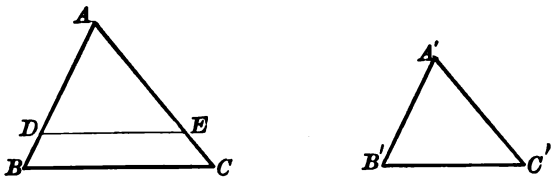


FIG. 187.

Given the $\triangle$s ABC and $A'B'C'$, with $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{AC}{A'C'}$.

To prove that $\triangle ABC \sim A'B'C'$.

Proof. 1. On AB lay off $AD=A'B'$, and on AC lay off $AE=A'C'$. Draw DE.

2. Then, since $\angle A$ is common,

and $\frac{AB}{AD}=\frac{AC}{AE}$, Why?

$\triangle ADE \sim \triangle ABC$. Why?

(It remains to prove that $\triangle A'B'C' \equiv \triangle ADE$.)

3. Now $\frac{BC}{DE}=\frac{AB}{AD}$. That is, $\frac{BC}{DE}=\frac{AB}{A'B'}$ Why?

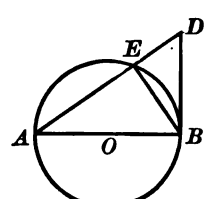
4. But $\frac{BC}{B'C'}=\frac{AB}{A'B'}$. Why?

5. $\therefore \frac{BC}{DE}=\frac{BC}{B'C'}$. Why?

6. $\therefore DE=B'C'$. Why?

(Let the student complete the proof.)

413. **EXERCISE**



1. AB is the diameter of a circle, BD is a tangent, and AD cuts the circle in E. Prove that $\triangle ABE \sim \triangle ABD$.

414. THEOREM. *In similar triangles, homologous altitudes have the same ratio as any two homologous sides.*

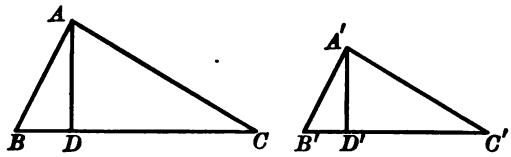


FIG. 188.

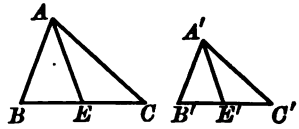
Given the similar triangles ABC and $A'B'C'$, with AD and $A'D'$ homologous altitudes.

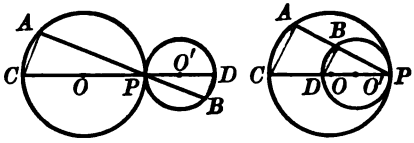
To prove that $\frac{AD}{A'D'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$.

SUGGESTION: rt. $\triangle ABD \sim$ rt. $\triangle A'B'D'$. Why?

415. EXERCISES

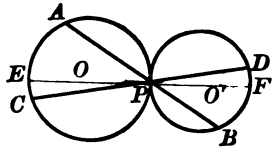
1. In similar triangles homologous medians have the same ratio as any two homologous sides. $\triangle ABE \sim \triangle A'B'E'$ (§ 410).

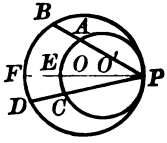




2. If two circles are tangent to each other, the chords formed by a straight line drawn through the point of contact have the same ratio as the diameters of the circles.

3. If two circles are tangent externally, the corresponding segments of two lines drawn through the point of contact and terminated by the circles are proportional. Apply Ex. 2.





4. If two circles are tangent internally at P any two chords of the greater circle through P are divided proportionally by the smaller circle.

416. THEOREM. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*

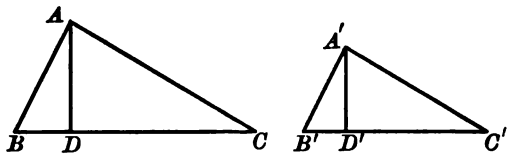


FIG. 189.

Given the similar $\triangle$s ABC and $A'B'C'$, with BC and $B'C'$ a pair of homologous sides.

To prove that $\dfrac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \dfrac{\overline{BC}^2}{\overline{B'C'}^2}$.

Proof. 1. Draw the altitudes AD and $A'D'$.

2. Then area $\triangle ABC = \frac{1}{2} BC \times AD$.
and area $\triangle A'B'C' = \frac{1}{2} B'C' \times A'D'$. Why?

3. $\therefore \dfrac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \dfrac{BC \times AD}{B'C' \times A'D'}$, Why?

or $= \dfrac{BC}{B'C'} \times \dfrac{AD}{A'D'}$. Why?

4. But $\dfrac{AD}{A'D'} = \dfrac{BC}{B'C'}$. Why?

5. $\therefore \dfrac{\text{area } \triangle ABC}{\text{area } \triangle A'B'C'} = \dfrac{BC}{B'C'} \times \dfrac{BC}{B'C'}$ or $\dfrac{\overline{BC}^2}{\overline{B'C'}^2}$. Why?

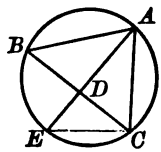
417. COROLLARY. *The areas of two similar triangles are to each other as the squares of any two homologous altitudes.*

418. **EXERCISES**

1. Through a point P on the side AB of a triangle ABC, a line is drawn parallel to BC so as to divide the triangle into two equal parts. Prove that $AB = AP\sqrt{2}$.

SUGGESTION: The small triangle formed is similar to $\triangle ABC$. Why?

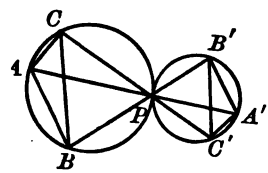
2. The area of an equilateral triangle circumscribed about a circle is four times the area of an inscribed equilateral triangle.



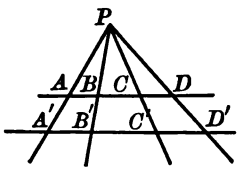
3. The bisector of an angle of a triangle intersects the opposite side in D and the circumscribed circle in E. Prove that $AB \times AC = AD \times AE$.

Draw EC, and show that $\triangle ABD \sim \triangle AEC$.

4. Two circles are tangent at P. Through P three lines are drawn, meeting one circle in A, B, C, and the other in A', B', C', respectively. Prove that $\triangle ABC$ and $A'B'C'$ are similar.



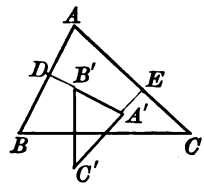
SUGGESTION: $\angle ACB = \angle APB$; and $\angle A'C'B' = \angle A'PB'$. Why?



5. If two parallel lines are cut by three or more transversals that pass through a common point, the corresponding segments of the parallels are proportional.

SUGGESTION: It is required to prove that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$. $\frac{AB}{A'B'} = \frac{PB}{PB'}$, and $\frac{BC}{B'C'} = \frac{PB}{PB'}$, etc.

6. The median drawn to any side of a triangle bisects any line segment parallel to that side and included between the other two sides.



7. Two triangles are similar if their sides are respectively perpendicular.

SUGGESTION: Show that $\triangle A'B'C'$ and $\angle B$ are each supplementary to $\angle DB'C'$.

8. The median drawn to the bases of a trapezoid (not a parallelogram) and the other two sides meet in a point if sufficiently produced.

9. If a point P is joined to the vertices of a triangle ABC, through any point E on PA a line parallel to AB is drawn, meeting PB in F; through F a line parallel to BC, meeting PC in G; and G is joined to E; the triangle EFG is similar to the triangle ABC.

419. Indirect measurement by proportion.

EXAMPLE: In the absence of more accurate surveying instruments, the height of an object, as a tree, may be estimated by the following method: Let the observer hold a staff in the vertical position AB, and sight along the line OD to the top of the tree, having an assistant mark the point B on the staff; then sight along the horizontal line OC, while the assistant marks the points A, C on the staff and tree, respectively. Now measure m, n, and d ($= OC$). The height x above the horizontal line may then be computed as follows:

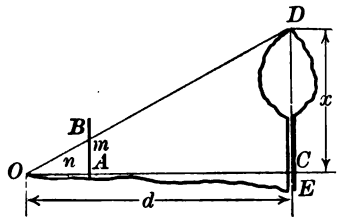


FIG. 190.

From similar $\triangle$ OAB and OCD,

$$\frac{x}{m}=\frac{d}{n}; \quad \text{or } \frac{x}{d}=\frac{m}{n}.$$

Find x, (1) if $d=50$ ft., $m=3$ ft., $n=2$ ft.; (2) if $d=60$ ft. and $\frac{m}{n}=\frac{3}{4}$; (3) if $d=80.5$ ft., and $\frac{m}{n}=.675$.

REMARK. It is important to observe that m and n *need not be measured in terms of the same unit as* d. All that is essential is that the number corresponding to the ratio $\frac{m}{n}$ be determined.

The principle involved in this example is the basis of all practical methods of indirect measurement. In more refined methods, instead of using a vertical staff AB to determine the ratio $\frac{m}{n}$, a transit (see p. 176), a sextant, or a similar instrument is used. By means of these instruments the $\angle$ COD is measured directly, and the ratio $\frac{m}{n}$, corresponding to this particular angle, is taken from tables in books specially prepared for this purpose. Thus if $\angle$ $COD=45°$, it can be easily shown that $\frac{m}{n}=1$. If $\angle$ $COD=60°$, $\frac{m}{n}=\sqrt{3}$, etc.

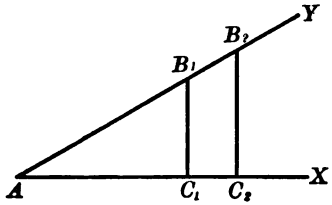


FIG. 191.

420. Trigonometric ratios. Functions of an acute angle. Let XAY be an acute angle. From points C_1, C_2 on AX draw perpendiculars to AX, meeting AY in B_1, B_2, respectively.

Then since $\triangle AC_1B_1 \sim \triangle AC_2B_2$

(1) $$\frac{B_1C_1}{AB_1} = \frac{B_2C_2}{AB_2};$$

(2) $$\frac{AC_1}{AB_1} = \frac{AC_2}{AB_2};$$

(3) $$\frac{B_1C_1}{AC_1} = \frac{B_2C_2}{AC_2}.$$

That is, the ratios (1), (2), (3) are constant (do not vary for different positions of C) for any particular angle XAY. These ratios are called *functions of the acute angle A*, and are distinguished by the names *sine, cosine,* and *tangent of angle A*, respectively.

Thus, referring to Fig. 192, in which the sides of the right triangle ACB are replaced by the corresponding small letters a, c, b;

$$\sin A = \frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}}.$$

$$\cos A = \frac{b}{c} = \frac{\text{adjacent leg}}{\text{hypotenuse}}.$$

$$\tan A = \frac{a}{b} = \frac{\text{opposite leg}}{\text{hypotenuse}}.$$

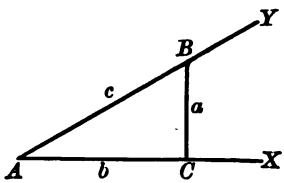


FIG. 192.

421. Functions of 45°, 60°, and 30°. In order that the student may clearly comprehend that the trigonometric ratios really have numerical values for any particular angle, we shall compute a table of functions for angles of 45°, 60°, and 30°.

1. Let XAY be an angle of 45°.

From a point C on AX draw a perpendicular to AX, meeting AY in B. Represent AC by n.

Then $CB = n$. Why?

$\therefore \overline{AB}^2 = 2n^2$, or $AB = n\sqrt{2}$. Why?

$$\therefore \sin 45° = \frac{a}{c} = \frac{n}{n\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2};$$

$$\cos 45° = \frac{b}{c} = \frac{n}{n\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2};$$

$$\tan 45° = \frac{a}{b} = \frac{n}{n} = 1.$$

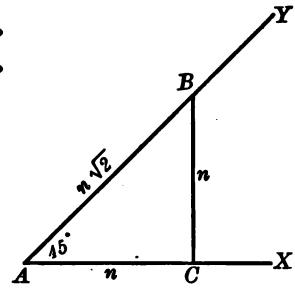


FIG. 193.

2. Let XAY be an angle of 60°, etc. Then $\angle ABC = 30°$. Why?

Represent AC by n. Then $AB = 2n$. Why?

$\therefore \overline{CB}^2 = (2n)^2 - n^2 = 3n^2$. Why? That is, $CB = n\sqrt{3}$.

$$\therefore \sin 60° = \sin A = \frac{a}{c} = \frac{n\sqrt{3}}{2n} = \frac{\sqrt{3}}{2};$$

$$\cos 60° = \cos A = \frac{b}{c} = \frac{n}{2n} = \frac{1}{2};$$

$$\tan 60° = \tan A = \frac{a}{b} = \frac{n\sqrt{3}}{n} = \sqrt{3}.$$

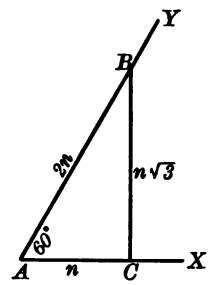


FIG. 194.

Moreover, since $\angle B = 30°$.

$$\sin 30° = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{n}{2n} = \frac{1}{2};$$

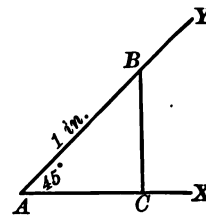
$$\cos 30° = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{n\sqrt{3}}{2n} = \frac{\sqrt{3}}{2};$$

$$\tan 30° = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{n}{n\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

FUNCTIONS OF 45°, 60°, AND 30° *

ANGLE	SIN	COS	TAN
45°	$\frac{\sqrt{2}}{2} = 0.7071+$	$\frac{\sqrt{2}}{2} = 0.7071+$	1.0000
60°	$\frac{\sqrt{3}}{2} = 0.8660+$	$\frac{1}{2} = 0.5000$	$\sqrt{3} = 1.7321+$
30°	$\frac{1}{2} = 0.5000$	$\frac{\sqrt{3}}{2} = 0.8660+$	$\frac{\sqrt{3}}{3} = 0.5774-$

422. **EXERCISES**



1. To find the functions of 45° by construction and measurement.

Construct an angle $XAY = 45°$. Lay off $AB = 1$ in., and draw $CB \perp AX$. Measure CB and AC to one hundredth of an inch. (Use a diagonal scale, Ex. 8, § 439.) Then, $\sin 45° = \frac{BC}{1}$; $\cos 45° = \frac{AC}{1}$; $\tan 45° = \frac{BC}{AC}$. Compare your results with the computed results above.

2. To find the functions of 60° by construction and measurement.

3. To find the functions of 30° by construction and measurement.

423. The transit. The figure shows a transit commonly used by surveyors for measuring angles. The instrument consists of the following essential parts: AB is a telescope giving an accurate line of vision, intersecting a horizontal axis EF, and a vertical axis GH, at right angles. The telescope AB may be turned about the axis EF, generating vertical angles, which may be read on a graduated circle C; it may also be turned about the axis GH, generating horizontal angles, which may be read on a graduated circle D. The vertical axis GH can be made plumb by suitable attachments.

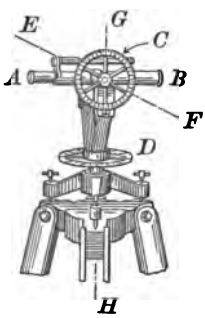


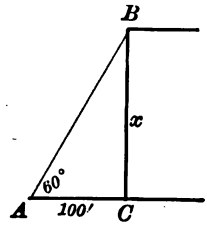
FIG. 195.

* For a more complete table of functions see page 218.

424. EXERCISES

The angle which a line from the eye to an object makes with a horizontal line in the same vertical plane is called an angle of *elevation*, or an angle of *depression*, according as the object is above or below the eye of the observer.

1. At a point 100 ft. in a horizontal line from the foot of a building, the angle of elevation of the top is observed to be 60°. Find the height of the building.

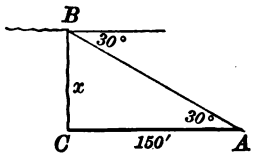


SOLUTION: In the rt. $\triangle ACB$, the unknown side x is *opposite* $\angle A$, and the known side 100 is *adjacent* to $\angle A$.

$\therefore \frac{x}{100} = \tan A = \tan 60° = 1.7321$ (see table).

$\therefore x = 173.2^+$ ft.

2. A and C are two points 500 feet apart on one bank of a straight river; B is a point on the other bank directly opposite C. If $\angle BAC = 60°$, find the width of the river.



3. From the top of a vertical cliff, the angle of depression of a point on the shore 150 feet from the base of the cliff is observed to be 30°. Find the height of the cliff. *Ans.* 86.6 ft.

4. From the top of a tower 120 feet high, the angle of depression of a house 40 feet high is observed to be 30°. How far is the top of the house from the tower? *Ans.* 138.6 feet.

5. A kite is held by a string 1200 feet long. If the string is assumed to be straight and it makes an angle of 45° with the horizontal, how high is the kite?

SUGGESTION: Employ sin 45°. *Ans.* 848.5 feet.

6. A ship has sailed due northeast a distance of 150 miles. How far is the ship north of the starting point? How far is it east of the starting point? *Ans.* 106.1 miles.

7. From the top of a mountain 2000 feet above sea level, the angle of depression of a distant ship is 30°. How far is the ship from the summit of the mountain? *Ans.* 4000 feet.

8. When a vertical pole casts a shadow exactly equal to its own length, what is the angle of elevation of the sun?

9. When a vertical post 17.3 feet high casts a shadow 10 feet long, what is the angle of elevation of the sun?

SUGGESTION: What is the tangent of the angle of elevation?

425. THEOREM. *Two similar polygons can be decomposed into the same number of triangles, similar each to each, and similarly placed.*

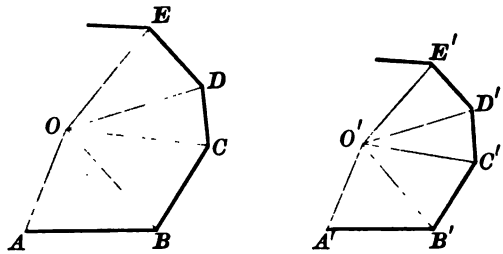


FIG. 196.

Given the similar polygons $ABCDE \cdots$ and $A'B'C'D'E' \cdots$.

To prove that the polygons can be divided into the same number of triangles, similar each to each, and similarly placed.

Proof. 1. Join any point O in $ABCDE \cdots$ to the vertices $A, B, C, \cdots$. Construct $\measuredangle B'A'O'$ and $A'B'O'$ equal to $\measuredangle BAO$ and ABO, respectively. Join O' to $A', B', C', \cdots$.

2. Then $\triangle ABO \sim \triangle A'B'O'$. Why?

3. $\therefore \dfrac{AB}{A'B'} = \dfrac{OB}{O'B'}$. Why?

4. But $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$. Why?

5. $\therefore \dfrac{OB}{O'B'} = \dfrac{BC}{B'C'}$. Why?

6. Also, since $\angle CBA = \angle C'B'A'$, Why?
$\angle CBO = \angle C'B'O'$. Why?

7. $\therefore \triangle BCO \sim \triangle B'C'O'$. Why?

In like manner, it can be proved that $\triangle CDO \sim \triangle C'D'O'$, etc.

Exercise. Prove § 425, by drawing diagonals from two corresponding vertices.

426. THEOREM. *If two polygons are composed of the same number of triangles, similar each to each, and similarly placed, the polygons are similar.*

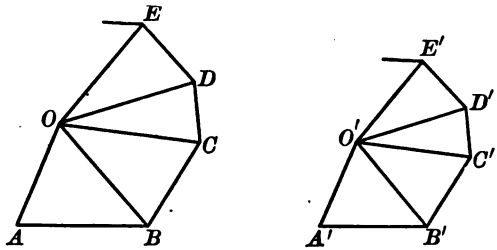


FIG. 197.

Given the polygons $ABCDE \cdots$ and $A'B'C'D'E' \cdots$ composed of the $\triangle$s OAB, OBC, OCD, $\cdots$; $O'A'B'$, $O'B'C'$, $O'C'D'$, $\cdots$, respectively similar and similarly placed.

To prove that the polygons $ABCDE \cdots$ and $A'B'C'D'E' \cdots$ are similar.

Proof. 1. $\angle OBA = \angle O'B'A'$, and $\angle CBO = C'B'O'$. Why?

$\therefore \angle CBA = \angle C'B'A'$.

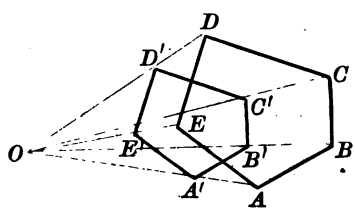
Similarly $\angle DCB = \angle D'C'B'$, etc.

2. Also $\dfrac{AB}{A'B'} = \left(\dfrac{OB}{O'B'}\right) = \dfrac{BC}{B'C'}$, etc. Why?

427. EXERCISES

1. Two regular polygons of the same number of sides are similar.

If the vertices A, B, C, D, E of a polygon are joined to a point O, and points A', B', C', D', E' on OA, OB, OC, OD, OE are joined successively, the polygons $ABCDE$ and $A'B'C'D'E'$ are said to be in *perspective;* and O is called the *center of perspective.*



2. If two polygons are in perspective and their sides are, respectively, parallel, the polygons are similar.

3. Any two similar polygons may be placed in perspective.

428. THEOREM. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*

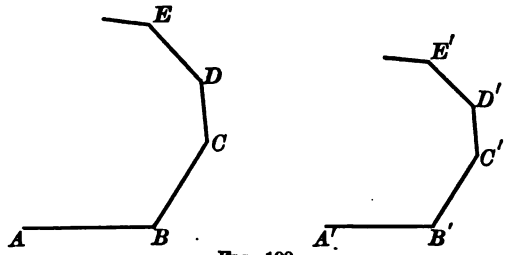


FIG. 198.

Given the similar polygons $ABCDE \cdots$ and $A'B'C'D'E' \cdots$ with perimeters p and p', respectively.

To prove that $$\frac{p}{p'} = \frac{AB}{A'B'}.$$

Proof. 1. $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \text{, etc.}$$ Why?

2. $$\therefore \frac{AB + BC + CD + \cdots}{A'B' + B'C' + C'D' + \cdots} = \frac{AB}{A'B'}.$$ § 381.

That is, $$\frac{p}{p'} = \frac{AB}{A'B'}.$$

429. EXERCISES

1. The perimeters of two similar polygons are 300 ft. and 400 ft. If a side of the first is 25 feet, find the homologous side of the second.

2. The bases of a trapezoid are 6 in. and 10 in. and the altitude is 2 in. Find the altitudes of the triangles formed by producing the sides till they meet.

3. The diameter of the moon is approximately 3600 miles and is approximately 240,000 miles distant from the earth. How far must a disk $\frac{1}{2}$ in. in diameter be held from the eye to just obscure a full moon?

4. I hold a ruler vertically before me and adjust its distance so that a certain space on the ruler subtends a space of 1 ft. on a graduated vertical rod at a distance of 100 ft. Keeping the ruler fixed in position, I observe that the same space on the ruler subtends a space of 3 ft. on the rod at a more distant point. How far is this point from my eye?

430. THEOREM. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*

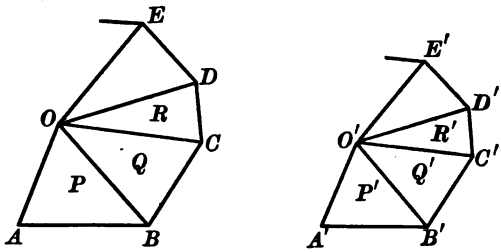


FIG. 199.

Given the similar polygons $ABCDE \cdots$ and $A'B'C'D'E' \cdots$ with areas S and S', respectively.

To prove that $$\frac{S}{S'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. 1. Decompose the polygons into similar triangles $P, Q, R \cdots$, and $P', Q', R' \cdots$, respectively. § 425.

2. Then $\frac{P}{P'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$; $\frac{Q}{Q'} = \frac{\overline{BC}^2}{\overline{B'C'}^2}$; $\frac{R}{R'} = \frac{\overline{CD}^2}{\overline{C'D'}^2}$, etc. Why?

3. But $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} =$, etc. Why?

$\therefore \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{CD}^2}{\overline{C'D'}^2} =$, etc. Why?

4. $\therefore \frac{P}{P'} = \frac{Q}{Q'} = \frac{R}{R'} =$, etc. Why?

5. $\therefore \frac{P+Q+R+\cdots}{P'+Q'+R'+\cdots} = \frac{P}{P'}$. Why?

6. $\therefore \frac{S}{S'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$. Why?

Exercise. An enlarged photograph is 9 times the size of the original. If a certain dimension of the former measures 4 in., what is the length of the corresponding dimension on the original?

CONSTRUCTION PROBLEMS

431. EXERCISES

1. Upon a given line segment as one side to construct a polygon similar to a given polygon. Apply § 426.

2. To construct a polygon similar to a given polygon, and having its perimeter equal to a given line segment.

SUGGESTION: Find the sides of the required polygon by § 402.

3. To inscribe in a given circle a triangle similar to a given triangle.

SUGGESTION: Circumscribe a circle about the given triangle and apply § 426.

4. To circumscribe about a given circle a polygon similar to a given regular polygon.

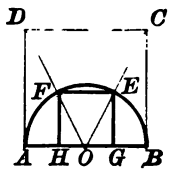
5. To inscribe in a given circle a polygon similar to a given regular polygon. Apply § 307, § 426.

6. To construct a polygon similar to, and equal to the sum of, two given similar polygons.

ANALYSIS. Let P and Q be the given polygons, and let S be the required polygon; also let a, b, and c be homologous sides, respectively. Then $\frac{P}{S} = \frac{a^2}{c^2}$, and $\frac{Q}{S} = \frac{b^2}{c^2}$. Why? $\therefore \frac{P+Q}{S} = \frac{a^2+b^2}{c^2}$ (addition). But $P + Q = S$ (by hypothesis). $\therefore a^2 + b^2 = c^2$. Why?

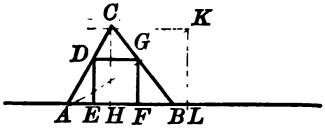
7. To construct a polygon similar to, and equal to the difference between two given similar polygons.

8. To inscribe a square in a given semicircle.



SUGGESTION: Construct the square $ABCD$, and join O to C and D, cutting the semicircle in E and F. Draw EG and FH perpendicular to AB. $HGEF$ is the required square.

9. To inscribe a square in a given triangle.



SUGGESTION: Construct the square $CHLK$, and join AK. G is one vertex of the required square.

10. To inscribe in a given semicircle a rectangle similar to a given rectangle.

11. To inscribe in a given triangle a rectangle similar to a given rectangle.

PROPORTIONAL LINE SEGMENTS

432. THEOREM. *If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.*

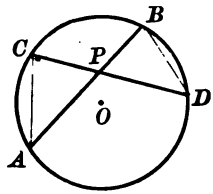


FIG. 200.

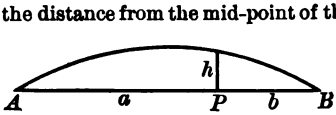
Given the circle with two chords AB and CD intersecting in P.

To prove that $AP \times BP = CP \times DP$.

SUGGESTION: Draw AC and BD.

433. **EXERCISES**

1. The distance between two points A, B, on a railway curve is 200 ft.; and the distance from the mid-point of the chord AB to the curve is 4 feet. Find the diameter. Base your solution on § 432.



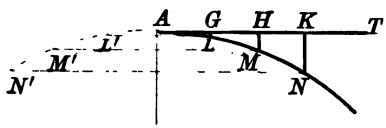
2. The following formula, used in railway location work, is a close approximation, based on the above theorem, for finding the perpendicular offsets from a chord to points on a curve:

Let P be any point on the chord AB, dividing it into segments a, b; let h be the perpendicular offset, and let R be the radius. Then if R is very large compared with the chord AB,

$$h = \frac{ab}{2R}, \text{ approximately.}$$

EXAMPLE: Given $a = 150$ ft., $b = 50$ ft., $R = 2500$ ft. Find h from the formula; also find the exact value of h and compare results.

3. In order to locate points on a proposed railway curve ABC, an engineer was forced, on account of obstructions, to locate the points



L, M, N by perpendicular offsets from the tangent AT. If the radius of the curve was 2500 feet, and $AG = GH = HK = 50$ ft., find the perpendicular offsets GL, HM, and KN.

4. Show how to compute the diameter of a fly wheel, given any segment of the rim. Is the formula of Ex. 2 applicable in this case?

434. A secant: A straight line which cuts a circle in two points is called a *secant.*

By *a secant from an external point to a circle* is meant the segment of the secant between the given external point and the farther point of intersection of the secant and the circle. The segment between the external point and the nearer point of intersection is called the *external segment* of the secant.

435. THEOREM. *If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.*

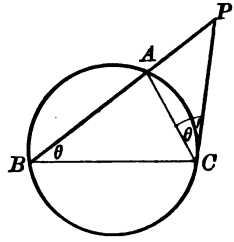


FIG. 201.

Given the circle with PC a tangent and PB a secant drawn from the external point P.

To prove that $$\frac{PB}{PC} = \frac{PC}{PA}.$$

SUGGESTION: Draw CA and CB. $\triangle$ PCB and PCA are similar.

436. COROLLARY. *If from a fixed point without a circle, a secant is drawn, the product of the secant and its external segment is constant.*

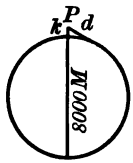
For $PB \times PA = \overline{PC}^2$ (a constant). Why?

437. EXERCISES

1. Find the length of the tangent from a point 9 in. distant from a circle whose radius is $3\frac{1}{2}$ in.

2. Find the radius of a circle if a tangent from a point 4 in. from the circle is 8 inches long.

3. A useful formula, in engineering, for estimating the greatest distance at which an object, as the light on a lighthouse of known elevation, is visible at sea is based on the above theorem.



Let k represent the height of an object on the earth's surface, and let d be the distance from the object to the horizon. Then, in the figure

$$\frac{8000 + k}{d} = \frac{d}{k}. \quad \therefore d^2 = k(8000 + k).$$

Now for all practical purposes k is very small compared with the diameter of the earth, and hence the k in the parentheses may be ignored without sensible error, giving

$$d^2 = 8000\,k. \quad \text{Or, } k = \frac{d^2}{8000},$$

where k and d are expressed in miles. To express k in feet, let $h = 5280\,k$. Then $k = \dfrac{h}{5280}$.

$$\therefore \frac{h}{5280} = \frac{d^2}{8000}. \quad \therefore h = \frac{528}{800}d^2. \quad \text{That is, } h = \frac{2}{3}d^2 \text{ (nearly).}$$

Hence, *the height in feet is equal to $\frac{2}{3}$ the square of the distance in miles.*

4. The maximum height which is deemed desirable for a lighthouse is 250 feet above sea level. How far is its light visible at sea?

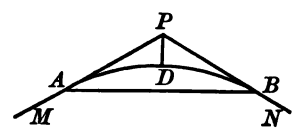
5. How high must an aeronaut rise in order to see 100 miles over a level plain?

6. How far can a man six feet high see over a level plain?

7. A lookout in the rigging of a vessel, 75 feet above the water, can just see a beacon light known to be 80 feet above the sea. How far is the vessel from the lighthouse?

8. The Woolworth building in New York City is 750 feet high. Assuming the base to be 35 feet above sea level, how far is a ship visible from the tower, if the highest point of the ship is 80 feet above the water?

9. MP and NP represent the center lines of two cement sidewalks. It is desired to join the walks by an arc of a circle, tangent at A and B, each 10 feet from P, and so that the distance of the arc from P shall be 4 feet. Find the radius of the circle.

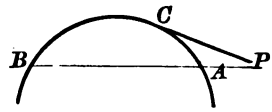


Is the formula of Ex. 3 applicable?

CONSTRUCTION PROBLEMS

438. EXERCISES

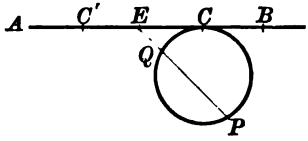
1. To draw a tangent to a circle from an external point, when the center is inaccessible.



ANALYSIS: Suppose PC is the required tangent. Draw a secant PB. Then $\frac{PB}{PC}=\frac{PC}{PA}$. (Why?)

Find PC by § 399.

2. To construct a circle which shall pass through two given points and be tangent to a given straight line.



ANALYSIS: Suppose the construction made; let the circle be tangent to AB at C (or C'). Draw PQ, and produce it to meet AB at E, if possible. Then $\frac{EQ}{EC}=\frac{EC}{EP}$. (Why?)

Find EC by § 399.

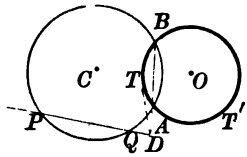
If PQ does not cut AB when produced, the construction is simple. Let the student explain.

In general, how many solutions are possible?

3. To construct a circle which shall pass through a given point and be tangent to two given straight lines.

SUGGESTION: Locate a second point through which the circle must pass by symmetry; then apply Ex. 2.

4. To construct a circle which shall pass through two given points and be tangent to a given circle.



SUGGESTION: Describe any circle through P and Q, cutting the given circle in A and B. Draw BA and produce it to meet PQ in D. Draw a tangent DT to the given circle. A circle passing through P, Q, and T will be tangent to the given circle.

In general, how many solutions are possible?

5. To construct a circle which shall be tangent to two given straight lines, and which shall also be tangent to a given circle.

6. Show how to construct the mean proportional between two line segments by using § 435.

7. To construct a parallelogram equal to a given square and having the difference between its base and altitude equal to a given segment.

439. MISCELLANEOUS EXERCISES

1. If two circles cut each other, tangents drawn to the two circles from any point on the common chord produced are equal.

2. If two circles cut each other, the common chord, if produced, bisects the common exterior tangents.

3. The point of intersection of the interior tangents to two circles divides the segment joining the centers of the circles into parts whose ratio is equal to the ratio of the radii.

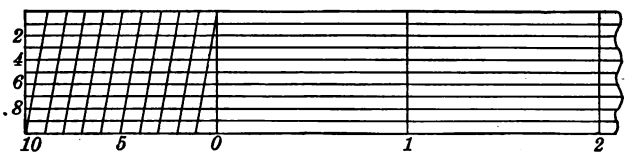
4. The point of intersection of the exterior tangents to two circles divides the segment joining the centers externally into segments whose ratio is equal to the ratio of the radii. See § 393.

5. If parallel diameters of two circles are drawn, straight lines drawn through their extremities, on the same side of the line joining the centers of the circle, pass through the intersection of the common exterior tangents; and straight lines drawn through their extremities on opposite sides of the line through the centers pass through the intersection of the common interior tangents.

6. Show how to draw the common tangents to two circles by using Ex. 5.

7. AD and BE are two altitudes of the triangle ABC. Show that the triangles CED and ABC are similar.

8. The figure represents a diagonal scale used by draftsmen. The distance from 0 to 10 is one inch.



Explain the principle involved in the use of the diagonal scale, and show how to set the dividers to the following distances: .25 in.; .77 in.; 1.98 in.; 1.03 in.

9. In two similar polygons, two homologous sides are 12 ft. and 20 ft. The area of the first polygon is 500 sq. ft. Find the area of the second polygon.

10. The line segment drawn through the intersection of the diagonals of a trapezoid parallel to the bases and terminating in the other two sides is bisected by the diagonals.

11. If three or more intersecting lines intercept proportional segments upon two parallel lines, they pass through a common point.

HINT: Use an indirect method. See § 418, Ex. 5.

12. If two circles are tangent to each other, their common exterior tangent is the mean proportional between their diameters.

SUGGESTION: Let AB be the common tangent. Draw the diameters AC and BD. Join the point of contact P of the circles to A, B, C, and D. Show that APD and BPC are straight lines perpendicular to each other, and that $\triangle CAB$ and ABD are similar.

13. A is the mid-point of CD, an arc of a circle; and AE is any chord intersecting the chord CD in P. Prove that $AP \times AE$ is a constant.

SUGGESTION: Draw the diameter AB, cutting CD in M; and draw BE. $\triangle AMP$ is similar to $\triangle ABE$.

14. If each of three circles cuts the other two, the common chords intersect in a common point. Apply § 432, or Ex. 1, § 439.

15. In an inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.

SUGGESTION: Let $ABCD$ be the inscribed quadrilateral. Draw AE, cutting the diagonal BD in E, and making $\angle EAD = \angle BAC$. $\triangle ABE \sim \triangle ACD$, and $\triangle AED \sim \triangle ABC$.

16. If a chord of a circle is bisected by another chord, each segment of the first chord is a mean proportional between the segments of the second chord.

17. If the line passing through the centers of two circles meets the common exterior tangents at O and the circles in the consecutive points A, B, C, D, then $OA \times OD = OB \times OC$.

18. The perpendicular from any point on a circle to a chord is the mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.

19. If two circles cut each other in the points P and Q, and through P any straight line is drawn terminating in the circles at A and B, the segments QA and BQ are to each other as the diameters of the circles.

20. To divide a line segment into two parts so that one part shall be a mean proportional between the other part and the whole segment.

ANALYSIS: Suppose a divided into the segments x and $a - x$, so that $\frac{a}{x} = \frac{x}{a - x}$.

Then $x^2 = a(a - x)$. $\therefore x^2 + ax = a^2$, or forming a proportion

$$\frac{a + x}{a} = \frac{a}{x}.$$

Reference to § 435 shows that the construction can be based on this theorem. Let a represent the diameter and tangent.

21. To divide a line segment into two parts so that the rectangle whose adjacent sides are equal to these segments shall be equal to a given square.

ANALYSIS: Let a be the given segment, and c the side of the given square. Suppose a divided into segments x and $a - x$ so that $x(a-x) = c^2$.

Reference to § 399 shows that the construction can be based on this problem.

22. To divide a line segment externally into two segments so that the rectangle whose adjacent sides are equal to these segments shall be equal to a given square.

ANALYSIS: Suppose a divided externally into segments x and $a + x$ so that $x(x + a) = c^2$. Then $\frac{x+a}{c} = \frac{c}{x}$. Compare with Ex. 20.

23. To divide a triangle into two equal parts by a line parallel to one of the sides.

24. To divide a triangle into two equal parts by a line drawn perpendicular to one of its sides.

25. To inscribe in a given semicircle a rectangle similar to a given rectangle.

26. To inscribe in a given triangle a parallelogram similar to a given parallelogram.

27. To divide a rectangle into two rectangles which shall have a given ratio $\frac{m}{n}$.

28. To construct a rectangle having a given ratio $\frac{m}{n}$ to a given rectangle.

29. To construct a square having a given ratio $\frac{m}{n}$ to a given square. Apply Ex. 28 and § 376.

30. Through a given point P within a given circle, to draw a chord AB so that the segments shall have a given ratio $\frac{m}{n}$.

ANALYSIS: Represent the segments of the required chord by x and y. Draw any chord through P and represent the segments by a and b. Then $xy = ab$, and $\frac{x}{y} = \frac{m}{n}$. (Why?) Eliminating y between these equations,

$$\frac{n}{m}x^2 = ab, \text{ or } x^2 = \frac{m}{n}ab.$$

Hence x is a side of the square having the ratio $\frac{m}{n}$ to the rectangle whose adjacent sides are a and b.

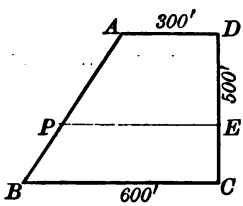
31. Through a given point P without a circle to draw a secant PAB so that the segments of the secant shall have a given ratio $\frac{m}{n}$.

SUGGESTION: Base your analysis on § 435 or § 436.

32. A boatman who can row 5 miles an hour in still water starts to row a boat directly across a stream a mile wide. If the stream flows at the rate of 3 miles an hour, how far down the stream will he land?

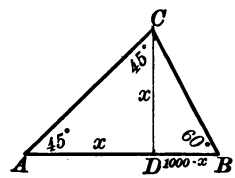
33. Suppose in the preceding problem the boatman wishes to row directly across the stream. How far up stream on the opposite side must he point his boat?

34. An aeroplane rises in a straight line at an angle of 30° with the horizontal at the rate of 50 miles an hour. How fast is it rising vertically? How fast is it moving horizontally?



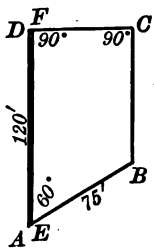
35. $ABCD$ represents a plot of ground with dimensions as shown; $AD \parallel BC$, and $CD \perp BC$. It is desired to divide $ABCD$ into two equal parts by a line $PE \parallel BC$. Required the lengths of AP and PE.

36. The angle of elevation of an aeroplane from a station due east of it is 60°, and from another station due west of the former and 1000 yards distant it is 45°. Find the height of the aeroplane.



Represent the required distance CD by x. Then $AD = x$. Why? Therefore $DB = 1000 - x$.

$\therefore \frac{x}{1000 - x} = \tan 60°$. *Ans.* 634 yards nearly.



37. P and Q are two points on opposite sides of a mountain; S is a point visible from P and Q; PS and QS are 8 miles and 12 miles, respectively, and the angle PSQ is 60°. Find the distance between P and Q.

Ans. 10.58 miles.

38. A man owns a city lot with the form and dimensions shown in the figure. He wishes to sell to his neighbor a strip $AEFD$ having a frontage AE equal to 1 foot. If the property is worth $2000, how much should he receive for the strip?

39. From the top of a cliff 100 feet high, the angle of depression of a ship at sea is observed to be 45°; in 4 seconds the angle of depression is again observed and found to be 30°. If the line of motion of the ship points directly to the cliff, how fast is the ship moving? *Ans.* 12.5 miles an hour nearly.

40. The angle of elevation of the top of a tree is 45° from a point on the ground in the same horizontal plane as the tree. From an upper window, 20 feet above this point on the ground, the angle of elevation is 30°. Find the height of the tree.

41. A 60-ft. pole stands on the top of a mound. If the angles of elevation of the top and bottom of the pole from a certain point are 60° and 45°, respectively, find the height of the mound above the point of observation.

42. A building is 130 feet high, and a flagstaff on the top of it is 60 feet high; 130 feet from the base of the building in a horizontal plane, the flagstaff subtends a certain angle. How far from the building along the same line is there another point at which the staff subtends the same angle?

43. The goal posts on a football field are $18\frac{1}{2}$ feet apart. If in making a touchdown the ball crosses the goal line 25 feet from the nearest goal post, how far back should it be carried so that the goal posts shall subtend the greatest possible angle from the point where the ball is placed?

CHAPTER VIII

METRIC RELATIONS

DERIVATION OF FORMULÆ

440. Constant and variable magnitudes. A geometric magnitude is called a *constant* or a *variable*, according as its value remains fixed, or changes, during a given discussion.

Thus consider the radius OP of a circle and its projection OQ on any diameter AB when the point P moves around the circle. It is clear that OP remains fixed in length, while OQ varies from a length equal to the radius when P is at B to zero when OP is perpendicular to AB; then increases again to a length equal to a radius when P is at A, etc. Hence, in this discussion OP is a constant and OQ is a variable.

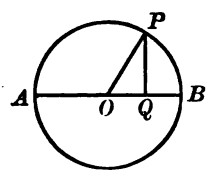


FIG. 202.

441. A function of a variable. If two variables are so related that a change in one produces a corresponding change in the other, each is called a *function* of the other.

Thus an angle and its tangent, sine, or cosine are so related that a change in the angle produces a corresponding change in each of the ratios mentioned; and conversely. Hence, the sine, cosine, and tangent of an angle are called functions of the angle.

NOTE. If a variable is a function of another variable, it should not be assumed that a certain change in the one produces an equal change in the other. For example, the area of a square is a function of the side (that is, if the side is changed, the area is changed); but a change in the side produces a much greater change in the area, comparatively. Thus, if the side is doubled, the area will be increased four times; if the side is increased 10 times, the area will be increased 100 times, etc.

This fact is of great importance in engineering. Doubling the size of a beam, for example, does not necessarily double the strength. Many of the great engineering disasters, such as the collapse of ponderous bridges, buildings, etc., have had their causes traced to mistakes of this nature.

442. A formula. A metric relation expressing a given variable as a function of one or more variables is called a *formula.*

Thus $s = ah$ is the formula expressing the area of a rectangle as a function (in terms) of two adjacent sides. In this formula s is said to be expressed *explicitly* in terms of a and h, and is called an *explicit function* of a and h; either a or h is called an *implicit function* of the other two.

To solve an equation for a given variable is to express this variable as an explicit function of the other variables involved.

Thus, if $s = ah$, then $a = \frac{s}{h}$. Here a is expressed explicitly in terms of s and h.

443. EXERCISES

Before substituting in a formula to find the numerical value of a variable corresponding to particular values of the other variables involved always express the unknown variable explicitly in terms of the other variables where possible.

1. Given $s = \frac{1}{2}bh$. Find b, when $s = 40$ and $h = 10$.

2. Given $A = \frac{1}{2}(a + b)h$: (*a*) Find h, when $A = 100$, $a = 8$, and $b = 12$; (*b*) find a, when $A = 120$, $b = 24$, and $h = 6$.

3. The base b of a triangle is increased by an amount equal to k. By how much must the altitude h be decreased so that the area may remain constant?

4. If a rectangle varies, keeping one side fixed, how does its area vary with respect to the adjacent side?

5. If a polygon varies in size, keeping the same shape, how does its area vary with respect to its sides?

6. If a man 6 feet high can see 3 miles on a smooth sea, can he see 100 miles from an elevation of 200 feet?

SUGGESTION: See Ex. 3, § 437.

7. The strength of a rectangular beam varies directly as the breadth multiplied by the square of its depth. Compare the strengh of a beam 4 in. in breadth and 3 in. in depth with that of a beam 3 in. in breadth and 4 in. in depth.

THE RIGHT TRIANGLE

444. THEOREM. *In a right triangle each leg is the mean proportional between the hypotenuse and its projection on the hypotenuse.*

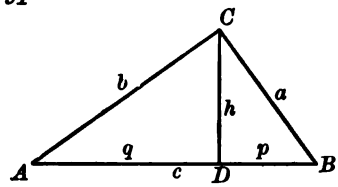


FIG. 203.

Given the right $\triangle ABC$ with the right angle at C, and p and q the projections of a and b on c, respectively.

To prove that $\frac{c}{a} = \frac{a}{p}$, and $\frac{c}{b} = \frac{b}{q}$.

Proof. 1. rt. $\triangle CDB \sim$ rt. $\triangle ABC$, and
rt. $\triangle CDA \sim$ rt. $\triangle ABC$. Why?

2. $\therefore \frac{c}{a} = \frac{a}{p}$, and $\frac{c}{b} = \frac{b}{q}$. Why?

445. COROLLARY 1. *In a right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.*

For, from step 2, $a^2 = cp$, and $b^2 = cq$. Why?

$\therefore a^2 + b^2 = cp + cq = c(p + q) = c^2$.

(Compare with § 366.)

446. COROLLARY 2. *In a right triangle the square of either leg is equal to the difference between the square of the hypotenuse and the square of the other leg.*

447. COROLLARY 3. *The altitude drawn to the hypotenuse of a right triangle divides the given triangle into two triangles which are similar to the given triangle and to each other.*

448. COROLLARY 4. *The altitude drawn to the hypotenuse of a right triangle is the mean proportional between the projections of the legs on the hypotenuse.*

For, since rt. $\triangle CDA \sim$ rt. $\triangle CDB$ (Cor. 3),

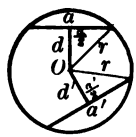
$$\frac{p}{h} = \frac{h}{q}. \qquad \text{Why?}$$

449. **EXERCISES**

1. If two chords of a circle are equal, they are equally distant from the center, and conversely; if they are unequal, the greater is the less distance from the center, and conversely.

SUGGESTION: $d^2 + \left(\frac{a}{2}\right)^2 = r^2 = d'^2 + \left(\frac{a'}{2}\right)^2$.

$$\therefore d^2 - d'^2 = \left(\frac{a'}{2}\right)^2 - \left(\frac{a}{2}\right)^2.$$



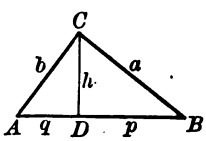
Compare d and d', if $a' = a$; if $a' > a$; if $a' < a$; and conversely.

2. The longest chord that can be drawn through a given point within a circle is the diameter drawn through that point; and the shortest chord is perpendicular to the diameter through that point.

3. A chord drawn from any point on a circle to either extremity of a diameter is the mean proportional between the diameter and its projection on the diameter.

4. The perpendicular from any point on a circle to a diameter is the mean proportional between the segments into which it divides the diameter.

5. The difference between the squares of two sides of any triangle is equal to the difference between the squares of the projections of these sides on the third side.



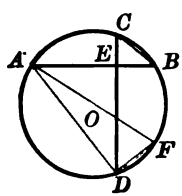
SUGGESTION: $b^2 - q^2 = h^2 = a^2 - p^2$. Why?

6. The squares of the two legs of a right triangle are to each other as their projections on the hypotenuse.

SUGGESTION: From step 2, § 444, $a^2 = cp$, and $b^2 = cq$. Why?

$$\therefore \frac{a^2}{b^2} = \frac{p}{q}. \qquad \text{Why?}$$

7. The squares of two chords drawn from the same point on a circle have the same ratio as the projections of the chords on the diameter drawn from the same point.



8. The sum of the squares of the segments of two perpendicular chords is equal to the square of the diameter of the circle.

SUGGESTION: Apply § 445 and prove $DF = BC$.

450. THEOREM. *In a right triangle:*

I. *Either leg is equal to the product of the hypotenuse into the sine of the opposite angle or the cosine of the adjacent angle.*

II. *Either leg is equal to the product of the other leg into the tangent of the angle opposite the former leg.*

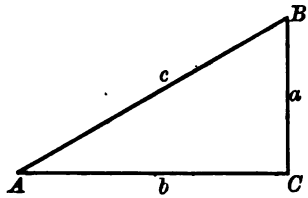


FIG. 204

Given the right $\triangle ACB$ with C the right angle.

To prove I. that $a = c \sin A = c \cos B$ or $b = c \sin B = c \cos A$. II. $a = b \tan A$ or $b = a \tan B$.

SUGGESTION: See definitions of sine, cosine, and tangent, § 420.

451. COROLLARY 1. *The sine of an acute angle is equal to the cosine of its complement.*

For $$\sin A = \frac{a}{c} = \cos B = \cos (90° - A).$$

452. COROLLARY 2. *The tangent of an acute angle is the reciprocal of the tangent of its complement.*

For $$\tan A = \frac{a}{b} = \frac{1}{\tan B} = \frac{1}{\tan (90° - A)}.$$

453. Use of a table of sines, cosines, and tangents. On page 218 is exhibited for reference a three-place table of sines, cosines, and tangents of acute angles at intervals of one degree. The use of the table may be described as consisting of two converse operations as follows: (1) Given an angle expressed in degrees and minutes to obtain its sine, cosine, or tangent from the table; and (2) given the sine, cosine, or tangent expressed as

a decimal fraction, to obtain, from the table, the corresponding angle, expressed in degrees and minutes.

I. Example (1). To obtain sin 42° 12′ from the table. We read directly that

$$\sin 42^\circ = .669$$

and

$$\sin 43^\circ = \underline{.682}$$
$$.013$$

∴ .013 corresponds to a difference of 1° or 60′.

∴ $\frac{12}{60} \times .013 = .003$ corresponds to a difference of 12′ (nearly).*

Since sin 43° *is greater than sin* 42°, .003 *must be added to* .669 *to give sin* 42° 12′. That is,

$$\sin 42^\circ 12' = .669 + .003 = .672.$$

Example (2). To obtain cos 43°12′ from the table. We read directly that

$$\cos 42^\circ = .743$$

and

$$\cos 43^\circ = \underline{.731}$$
$$.012$$

∴ .012 corresponds to a difference of 1° or 60′.

∴ $\frac{12}{60} \times .012 = .001$ corresponds to a difference of 12′.*

Since cos 43° *is less than cos* 42°, .001 *must be subtracted from* .743 *to give cos* 42° 12′. That is,

$$\cos 42^\circ 12' = .743 - .001 = .742.$$

The process for reading the tangent from the table is similar to the process for reading the sine and cosine.

II. Example (1). Given $\sin x = .524$, to obtain x from the table. From the column under sin we observe that .524 lies between

$$\sin 31^\circ = .515$$

and

$$\sin 32^\circ = \underline{.530}$$
$$.015$$

∴ 1° or 60′ corresponds to a difference of .015.*

But

$$\sin x - \sin 31^\circ = \left\{ \begin{array}{r} .524 \\ \underline{-\ .515} \\ .009 \end{array} \right.$$

∴ this difference corresponds to $\frac{.009}{.015} \times 60' = 36'$.

$$\therefore x = 31^\circ\, 36'.$$

* For practical purposes small differences in the function are assumed to be proportional to the corresponding differences in the angle.

Example (2). Given $\cos y = .473$, to obtain y from the table. From the column under cos, we observe that .473 lies between

$$\cos 61^\circ = .485$$

and

$$\cos 62^\circ = \underline{.469}$$

$$.016$$

$\therefore$ 1° or 60′ corresponds to a difference of .016.

But

$$\cos 61^\circ - \cos y = \begin{cases} .485 \\ \underline{.473} \\ .012 \end{cases}$$

$\therefore$ this difference corresponds to $\frac{.012}{.016} \times 60' = 45'$.

$$\therefore x = 61^\circ 45'.$$

454. Solution of the right triangle. *To solve a right triangle* is to compute the unknown parts when certain determining parts are known. See § 219.

455. EXERCISES

In lettering a right triangle ABC it is customary to represent the vertex of the right angle by C. Then a and b are the legs opposite the angles A and B, respectively, and c is the hypotenuse.

Solve the following right triangles:

1. Given $b=2.70$, $A=54^\circ$. *Ans.* $B=36^\circ$, $a=3.72$, $c=4.59$.

SOLUTION: $B=90^\circ-A=36^\circ$.

$a=b \tan A$, § 450, II $\qquad c=\frac{b}{\cos A}$, § 450, I

$=2.7\times 1.38$ $\qquad =\frac{2.7}{.588}$

$=3.72.$ $\qquad =4.59.$

2. Given $c=140$, $B=24^\circ 12'$. *Ans.* $A=65^\circ 48'$, $a=128$, $b=57.4$.

SOLUTION: $A=90^\circ-B=65^\circ 48'$.

$a=c\cos B$, § 450 I $\qquad b=c\sin B$, § 450 I

$=140\times .912$ $\qquad =140\times .410$

$=128.$ $\qquad =57.4.$

3. Given $a=3.4$, $b=2.6$. *Ans.* $c=4.3$, $A=52^\circ 36'$, $B=37^\circ 24'$.

SOLUTION: $\tan A=\frac{a}{b}$, by Def. $\qquad c=\sqrt{a^2+b^2}$, § 445

$=\frac{3.4}{2.6}$ $\qquad =\sqrt{(3.4)^2+(2.6)^2}$

$=1.31.$

$\therefore A=52^\circ 36'.$ $\qquad =4.3.$

$B=90^\circ-A=37^\circ 24'.$

4. Given $b = 42.4$, $A = 32° 14'$. *Ans.* $B = 57° 46'$, $a = 26.7$, $c = 50.1$.

5. Given $c = 3.40$, $B = 29°$. *Ans.* $A = 61°$, $b = 1.65$, $a = 2.98$.

6. Given $c = 26$, $A = 37° 42'$. *Ans.* $B = 52° 18'$, $a = 15.9$, $b = 20.2$.

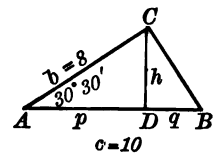
7. Given $a = 121$, $b = 37$. *Ans.* $A = 73°$, $B = 17°$, $c = 127$.

8. Given $c = 5.4$, $a = 2.6$. *Ans.* $A = 28° 48'$, $B = 61° 12'$, $b = 4.7$.

SUGGESTION: $\sin A = \frac{a}{c}$; $b = \sqrt{c^2 - a^2}$.

9. Given $c = 63$, $b = 47$.
Ans. $A = 41° 45'$, $B = 48° 15'$, $a = 42$.

10. In the triangle ABC, $AC = 8$, $AB = 10$, $\angle A = 30° 30'$. Find p and q, the projections of AC and CB on AB; also find h, the altitude on AB.



SUGGESTION: In the rt. $\triangle ADC$,
$p = 8 \cos 30° 30'$; and $h = 8 \sin 30° 30'$.

11. In the triangle ABC, $AC = 16$, $AB = 24$, $\angle A = 48° 15'$. Find the remaining side of the triangle and the area.

12. The length of a lake subtends at a certain point an angle of $75° 20'$, and the distance from this point to the two extremities of the lake are 350 feet and 540 feet. Find the length of the lake.

456. Directed line segments. Hitherto we have considered a line segment, as OA, to have length only; its measure has always been a positive number. It is often of the greatest convenience, however, to make use in geometry of the algebraic notion of positive and negative numbers. This is possible whenever it becomes desirable to distinguish between the two opposite directions or senses which exist on a line. Let O be any point on a line. The position of any other point, as A or A', is known as soon as we know *its distance from* O and *its direction from* O. Thus we may say that A is "2 in. to the right of O" or that "A' is 2 in. to the left of O." If we agree to consider segments measured to the right as positive and segments measured to the left as negative, we may also say that A is $+2$ in. from O and that A' is -2 in. from O. The most familiar example of this use of directed segments is perhaps found in the thermometer scale, where $+10°$ and $-10°$ are familiar expressions to every one.

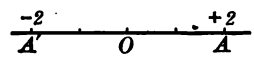


FIG. 205.

457. Sine, cosine, and tangent of an obtuse angle. Let CAB be an obtuse angle. Through a point B_1 on AB draw a perpendicular C_1B_1 to the line AC. Since CAB is obtuse, C_1 will fall on CA produced. The segment AC_1 is, then, regarded as negative (§ 456).

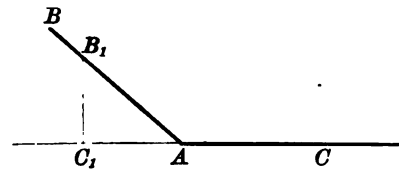


FIG. 206.

The sine, cosine, and tangent of CAB are now defined by the following ratios:

$$\sin \angle CAB = \frac{C_1B_1}{AB_1}, \cos \angle CAB = \frac{AC_1}{AB_1}, \tan \angle CAB = \frac{C_1B_1}{AC_1}.$$

Since AC_1 is regarded as negative, while C_1B_1 and AB_1 are regarded as positive, *the sine of any obtuse angle is a positive number* and *the cosine and tangent of any obtuse angle are both negative numbers.* It follows, therefore, that (1) *the sine of the obtuse angle CAB is identically equal to the sine of the supplementary acute angle BAC_1; while* (2) *the cosine and the tangent of the obtuse angle CAB are numerically equal to the cosine and tangent respectively of the supplementary acute angle BAC_1, but opposite in sign.*

Hence, if A is any obtuse angle, we have

$$\sin A = \sin (2R - A),$$
$$\cos A = -\cos (2R - A),$$
$$\tan A = -\tan (2R - A).$$

EXAMPLE: To find the sine, cosine, and tangent of 135°.

SOLUTION: $\sin 135° = \sin (180° - 135°) = \sin 45° = \frac{\sqrt{2}}{2}.$

$$\cos 135° = -\cos (180° - 135°) = -\cos 45° = -\frac{\sqrt{2}}{2}.$$

$$\tan 135° - \tan (180° - 135°) = -\tan 45° = -1.$$

The student should construct the figure defining the functions of 135° and verify the values of sin 135°, cos 135°, tan 135° directly from the definition.

Exercise. Find the sine, cosine and tangent of 120°; of 150°.

458. Relation between the sine and cosine of an angle.

Let XAY represent any acute angle. From any point B on AY draw BC perpendicular to AX cutting AX in C.

Then in the right triangle ACB,

$$\sin A = \frac{a}{c}, \text{ and } \cos A = \frac{b}{c} \text{ (Def.).}$$

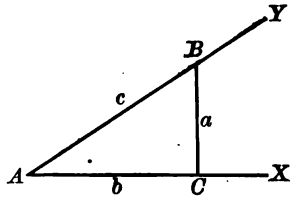


FIG. 207.

Squaring and adding these equations

$$(\sin A)^2 + (\cos A)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2 + b^2}{c^2}.$$

But $$a^2 + b^2 = c^2 \text{ (§ 445).}$$

$$\therefore (\sin A)^2 + (\cos A)^2 = 1.$$

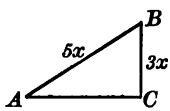
Moreover, since the sine and cosine of an obtuse angle are numerically equal to the sine and cosine of its supplement, respectively, § 457, the same relation holds for obtuse angles. Hence:

The sum of the squares of the sine and cosine of an (oblique) angle is equal to unity.

459. EXERCISES

1. Given A an acute angle and $\sin A = \frac{4}{5}$; find $\cos A$ and $\tan A$.

SOLUTION: In the right $\triangle ABC$,
let $BC = 3x$, and $AB = 5x$.
Thus $AC = \sqrt{(5x)^2 - (3x)^2} = 4x$.
$\therefore \cos A = \frac{4x}{5x} = \frac{4}{5}$; $\tan A = \frac{3x}{4x} = \frac{3}{4}$.



2. Given $\sin A = \frac{5}{13}$; A being an obtuse angle, find $\cos A$ and $\tan A$.
3. Given $\cos A = .6$; find $\sin A$ and $\tan A$. *Ans.* .8, 1.33.
4. Given $\tan A = \frac{15}{8}$; find $\sin A$ and $\cos A$. *Ans.* $\frac{15}{17}$, $\frac{8}{17}$.
5. Given $\cos A = \frac{7}{25}$; find $\sin A$ and $\tan A$. *Ans.* $\frac{24}{25}$, $\frac{24}{7}$.
6. Show that for oblique angles $(\tan A)^2 + 1 = \frac{1}{(\cos A)^2}$.
7. A flagstaff 80 feet high, on a horizontal plane, casts a shadow of 110 feet. Find the altitude (angle of elevation) of the sun.

SUGGESTION: Use the tangent.

OBLIQUE TRIANGLES

460. THEOREM. *In any triangle the three sides are proportional to the sines of the opposite angles.*

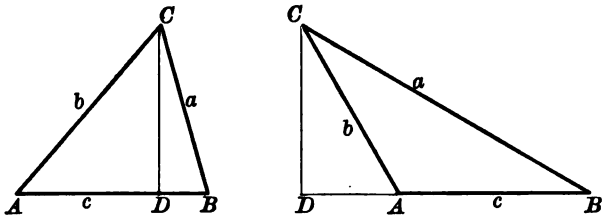


FIG. 208.

Given the triangle ABC.

To prove $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Proof. 1. Draw the altitude CD.

2. Then $CD = a \sin B = b \sin A$. § 450, I.

3. $$\therefore \frac{a}{\sin A}=\frac{b}{\sin B}.$$

In like manner by drawing the altitude on the side BC it can be proved that

$$\frac{b}{\sin B}=\frac{c}{\sin C}.$$

461. The law of sines. The preceding theorem is called the *law of sines.* By means of this law any triangle may be solved, (1) *when two angles and a side are known;* and (2) *when two sides and an angle opposite one of them are known.*

NOTE. In case (2) two solutions are possible. This results from the fact that the angles are computed from the sine, and supplementary angles have the same sine, § 457.

462. EXERCISES

1. Solve the triangle ABC, given $a = 50.0$, $A = 10° 12'$, $B = 46°, 36'$.
Ans. $C = 123° 12'$, $b = 205.1$, $c = 236.4$.

2. Solve the triangle ABC, given $c = 5.0$, $A = 58°$, $C = 57°$.

463. THEOREM. *In any triangle the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these sides and the cosine of the included angle.*

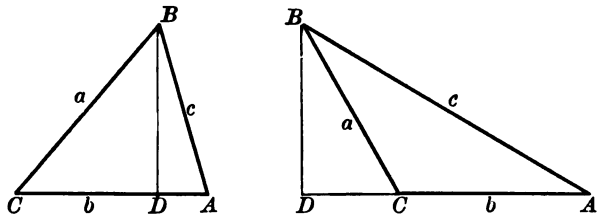


FIG. 209.

Given the triangle ABC.

To prove that $c^2 = a^2 + b^2 - 2\,ab \cos C$.

Proof. 1. Draw the altitude BD.

2. Then in the right $\triangle ABD$, $c^2 = \overline{BD}^2 + \overline{AD}^2$. Why?

3. But $BD = a \sin C$, Why?

and $AD = b - CD = b - a \cos C$. Why?

(Observe that CD is negative when $\angle C$ is obtuse.)

4. $\therefore c^2 = (a \sin C)^2 + (b - a \cos C)^2$ Why?

$= a^2(\sin C)^2 + b^2 - 2\,ab \cos C + a^2(\cos C)^2$

$= a^2\{(\sin C)^2 + (\cos C)^2\} + b^2 - 2\,ab \cos C.$

5. $\therefore c^2 = a^2 + b^2 - 2\,ab \cos C.$ § 458

In like manner by drawing the altitudes on the other two sides, it can be proved that

$$a^2 = b^2 + c^2 - 2\,bc \cos A;\ b^2 = c^2 + a^2 - 2\,ac \cos B.$$

464. The law of cosines. The theorem of § 463 is called the *law of cosines.* By means of this law the angles of a triangle may be computed when the three sides are known. For,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc};\ \cos B = \frac{c^2 + a^2 - b^2}{2\,ca};\ \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

Exercise. If the sides of a triangle are 7, 9, 11, find the greatest angle.

SUGGESTION: Let $a = 7$, $b = 9$, $c = 11$. Cos $C = \dfrac{a^2 + b^2 - c^2}{2\,ab}$, § 464.

COMPUTATION PROBLEMS

465. PROBLEM. *To find the altitude of a triangle, given the three sides, a, b, c.*

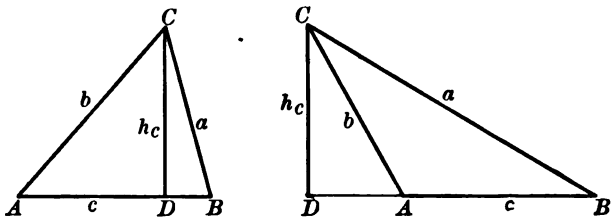


FIG. 210.

SOLUTION: 1. Draw the altitude $CD = h_c$.

2. In the right $\triangle ADC$

$$h_c^2 = b^2 - \overline{AD}^2. \qquad \text{Why ?}$$

3. But $$AD = b \cos A = b \cdot \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{b^2 + c^2 - a^2}{2\,c}. \qquad \S\ 464$$

4. $$\therefore h_c^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4\,c^2}$$

$$= \frac{4\,b^2c^2 - (b^2 + c^2 - a^2)^2}{4\,c^2}$$

$$= \frac{\{2\,bc + (b^2 + c^2 - a^2)\}\{2\,bc - (b^2 + c^2 - a^2)\}}{4\,c^2}$$

$$= \frac{\{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}}{4\,c^2}$$

$$= \frac{(b + c + a)(b + c - a)(a - b + c)(a + b - c)}{4\,c^2}.$$

5. Let $a + b + c = 2\,s.$

Then $b + c - a = 2\,s - 2\,a = 2(s - a).$

$a - b + c = 2\,s - 2\,b = 2(s - b).$

$a + b - c = 2\,s - 2\,c = 2(s - c).$

6. $$\therefore h_c^2 = \frac{2\,s \cdot 2(s - a) \cdot 2(s - b) \cdot 2(s - c)}{4\,c^2}$$

$$= \frac{4\,s(s - a)(s - b)(s - c)}{c^2}.$$

7. $$\therefore h_c = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}.$$

466. PROBLEM. *Given a, b, c, the sides of a triangle, to find the area.*

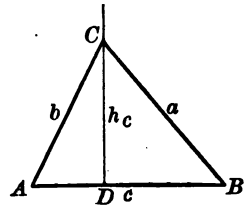


FIG. 211.

SOLUTION: 1. Draw the altitude h_c, and denote the area by S.

2. Then

$$S = \frac{1}{2}ch_c = \frac{1}{2}c \times \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

3. Hence $\quad S = \sqrt{s(s-a)(s-b)(s-c)}.$

467. PROBLEM. *Given a, b, c, the sides of a triangle, to find sin A, sin B, and sin C.*

SOLUTION: 1. $\text{Sin } A = \dfrac{h_c}{b}$

$$= \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)} \div b = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}.$$

2. Hence $\sin A = \dfrac{2S}{bc}$. Similarly, $\sin B = \dfrac{2S}{ac}$; $\sin C = \dfrac{2S}{ab}$.

468. EXERCISES

1. If two sides of a triangle are each equal to 8 and the included angle is 60°, find the third side.

SUGGESTION: Apply the law of cosines, § 463.

2. If two sides of a triangle are each equal to 10 and the included angle is 120°, find the third side.

3. If the sides of a triangle are 7, 9, 12, find the greatest angle.

4. If the sides of a triangle are 6, 7, 9, find the three altitudes.

5. If the sides of a triangle are 3 ft., 4 ft., and 6 ft., find the area, and also the three angles.

SUGGESTION: Apply §§ 464 and 466.

6. If b and c are two sides of a triangle and A is the included angle, show that its area is $\frac{1}{2}bc \sin A$.

SUGGESTION: $h_c = b \sin A$.

7. Find the areas of the triangles in Ex. 1 and Ex. 2 by Ex. 7.

8. If a is the side of an equilateral triangle, show that its area is $\dfrac{a^2}{4}\sqrt{3}$.

9. $AB = 500$ ft. is a base line along the bank of a river, and C is an object on the opposite bank. If $\angle BAC = 57°$, and $\angle CBA = 80° 20'$, find the breadth of the river.

SUGGESTION: Apply the law of sines.

469. PROBLEM. *Given a, b, c, the sides of a triangle, to find the radius R of the circumscribed circle.*

SOLUTION: 1. Draw the diameter CD and join BD.

2. Then $\triangle CDB$ is a rt. $\triangle$ and CD is the hypotenuse. Why?

3. Also $\angle A = \angle CDB$. Why?

4. $\therefore a = CD \sin \angle CDB = 2R \sin A$. Why?

5. $\therefore R = \dfrac{a}{2 \sin A}$; or $R = a \div \dfrac{4S}{bc} = \dfrac{abc}{4S}$,

(solving for R and substituting the value of $\sin A = \dfrac{2S}{bc}$, § 467).

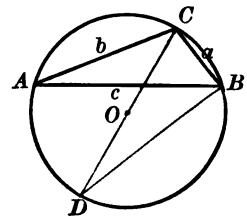


FIG. 212.

470. PROBLEM. *Given a, b, c, the sides of a triangle, to find the radius r of the inscribed circle.*

SOLUTION: 1. Join O, the center of the inscribed circle, to A, B, C, and draw the radii OD, OE, OF to the points of tangency on AB, BC, CA, respectively.

2. Then $\triangle ABC \equiv \triangle AOB + \triangle BOC + \triangle COA$.

3. $\therefore S = \frac{1}{2} rc + \frac{1}{2} ra + \frac{1}{2} rb = \frac{1}{2} r(a+b+c)$.

4. $\therefore r = \dfrac{2S}{a+b+c} = \dfrac{S}{s}$,

(solving for r and replacing $a+b+c$ by $2s$).

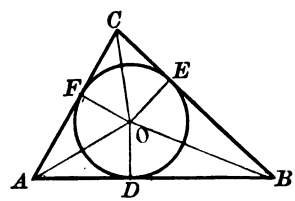


FIG. 213.

471. EXERCISES

1. If the sides of a triangle are 5 ft., 12 ft., and 13 ft., find the area of the triangle; and also the radii of the circumscribed and inscribed circles.

2. If the sides of a triangle are 5, 5, 8, find the area, the angles, and the radii of the circumscribed and the inscribed circles.

3. Given a, b, two adjacent sides of a parallelogram, and A, the included angle. Show that the area of the parallelogram is $ab \sin A$.

SUGGESTION: Find the altitude on one side.

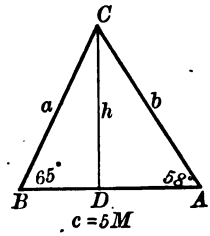
4. If two sides of a parallelogram are 4 ft. and 6 ft., and the included angle is 60°, find the diagonals and the area. Apply the law of cosines.

5. A ladder 40 feet long reaches a window 33 feet high, on one side of a street. When turned over upon its foot, it reaches another window 21 feet high, on the opposite side of the street. Find the width of the street.

6. Given k, l, the diagonals of a parallelogram, and α, the included angle. Show that the area of the parallelogram is $\frac{1}{2}\, kl \sin \alpha$.

7. If the diagonals of a parallelogram are 24 ft. and 32 ft., and an included angle is 45°, find the sides and the area.

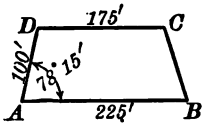
8. Two observers 5.0 miles apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 58° and 65°, respectively. Find the distances from the balloon to the observers, and also the height of the balloon above the plain.



Ans. Distances from observers, 5.1 m. and 5.4 m.; height above plain, 4.6 m.

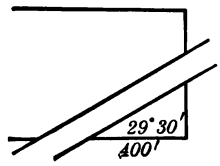
9. A lighthouse was observed from a ship to bear N. 35° E.; after the ship sailed due south 4 miles, it bore N. 18° E. Find the distance from the lighthouse to the ship in both positions.

NOTE. The phrase "to bear N. 35° E." means that the line of sight from the ship to the lighthouse is to the east of a line running due north from the ship, and makes with the latter line an angle of 35°.

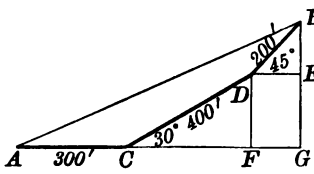


10. $ABCD$ is a plat of ground in the shape of a trapezoid with AB and DC the bases. Find the area, if $AB = 225$ ft., $AD = 100$ ft., $DC = 175$ ft., and $\angle BAD = 78° 15'$.

11. A railroad having a hundred feet right of way cuts through a farmer's field as shown in the figure. If the field is rectangular, and the measurements made are as shown, find the number of square rods occupied by the right of way; also find the assessed damages if the land is appraised at $100 an acre.



12. In a preliminary railroad survey the distance between two points A and B on opposite sides of an impenetrable swamp and the line of direction of AB were computed from the following data: Two points C and D visible from each other, and visible from A and B, respectively, were chosen, and distances and angles measured as shown.



Find the distance AB; also $\measuredangle CAB$ and ABD.

SUGGESTION: 1. Draw DF and $BG \perp AC$, produced, and draw $DE \parallel AC$.

2. Find $DF = EG$, BE, $DE = FG$, and CF.

3. Then in rt. $\triangle AGB$, AG and BG are known.

REGULAR POLYGONS AND CIRCLES

472. PROBLEM. *Given R, the radius of a circle, to find the side of an inscribed equilateral triangle.*

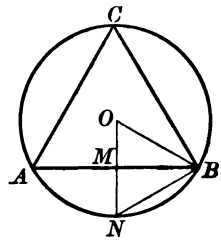


FIG. 214.

SOLUTION: 1. Let $AB = a$ be a side of the inscribed equilateral triangle. Join the center O to B, and draw $OM \perp AB$, cutting it in M.

2. Then $OB = R$, $OM = \frac{R}{2}$, $MB = \frac{a}{2}$. Why?

3. But $\overline{OB}^2 = \overline{OM}^2 + \overline{MB}^2$, or $R^2 = \left(\frac{R}{2}\right)^2 + \left(\frac{a}{2}\right)^2$. Why?

4. $\therefore a^2 = 4R^2 - R^2 = 3R^2$,

or $a = R\sqrt{3} = 1.73205080 \cdots R$.

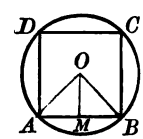
473. COROLLARY. *The side of a regular hexagon inscribed in a circle is equal to the radius.*

474. EXERCISES

1. If the radius of a circle is 1, find the side, the apothem, and the area of an inscribed equilateral triangle.

2. If the side of an equilateral triangle is a, show that the radius of the circumscribed circle is $\frac{a}{\sqrt{3}} = \frac{1}{3}a\sqrt{3}$.

3. If the radius of a circle is R, show that the side of an inscribed square is $R\sqrt{2}$.



SUGGESTION: Let $AB = a$ be a side of the inscribed square. Join OA and OB. $\triangle AOB$ is a right triangle. Why?

4. If the radius of a circle is R, show that the area of the inscribed square is $2R^2$, and the apothem is $\frac{R}{2}\sqrt{2}$.

475. PROBLEM. *Given R, the radius of a circle, to find the side of an inscribed regular dodecagon.*

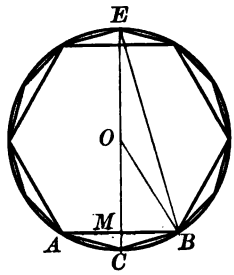


FIG. 215.

SOLUTION: 1. Let AB be a side of an inscribed regular hexagon. Draw the diameter CE, bisecting arc AB, and cutting AB in M. Join BC, BO, and BE.

2. Then BC is a side of a regular dodecagon. Why?

3. Since $\angle EBC$ is a right angle, Why?

$$\frac{CE}{BC} = \frac{BC}{CM}. \qquad \S\,444$$

$\therefore \overline{BC}^2 = CE \times CM = 2R \times (R - OM)$. Why?

It remains to find OM in terms of R.

4. Now $\overline{OM}^2 = \overline{OB}^2 - \overline{MB}^2 = R^2 - \left(\frac{R}{2}\right)^2 = \frac{3R^2}{4}$. Why?

$$\therefore OM = \frac{R}{2}\sqrt{3}.$$

5. $$\therefore \overline{BC}^2 = 2R \times \left(R - \frac{R}{2}\sqrt{3}\right) = R^2(2 - \sqrt{3}),$$

$$\text{or } BC = R\sqrt{2 - \sqrt{3}} = .51763809 \cdots R.$$

REMARK. By an exactly similar method, when the side of *any* inscribed regular polygon is known, the side of an inscribed regular polygon of double the number of sides can be found.

476. EXERCISES

1. If the radius of a circle is 1, find the side, the perimeter, and the area of an inscribed regular dodecagon.

2. Find the side of a regular octagon inscribed in a circle, if (1) the radius is 1; (2) if the radius is R.

477. PROBLEM. *Given R, the radius of a circle, to find the side of a circumscribed regular hexagon.*

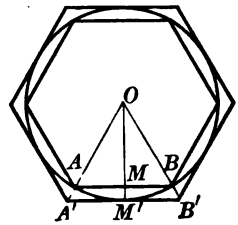


FIG. 216.

SOLUTION: 1. Let AB be a side of an inscribed regular hexagon. Draw $OM' \perp AB$, and draw $A'B'$ tangent to the circle at M', intersecting OA and OB, produced, at A' and B', respectively.

2. Then $A'B'$ is a side of a circumscribed regular hexagon. Why?

3. Now $\triangle AOB \sim \triangle A'OB'$. Why?

4. $\therefore \dfrac{A'B'}{AB} = \dfrac{OM'}{OM}$. Why?

Or $A'B' = \dfrac{OM'}{OM} \times AB$.

5. But $OM' = R$, $AB = R$, and $OM = \dfrac{R}{2}\sqrt{3}$. Why?

6. $\therefore A'B' = R^2 \div \dfrac{R}{2}\sqrt{3} = \dfrac{2R}{3}\sqrt{3} = 1.15470053 \cdots R$.

REMARK. By an exactly similar process, when the side of any inscribed regular polygon is known, the side of a circumscribed regular polygon of the same number of sides can be found.

478. EXERCISES

1. If the radius of a circle is 1, find the perimeter and the area of a circumscribed regular hexagon.

2. If the radius of a circle is R, find the side of a circumscribed equilateral triangle.

3. If the radius of a circle is 1, find the perimeter and the area of a circumscribed regular dodecagon. Use the results of Ex. 1, § 476.

4. If the radius of a circle is 1, find the perimeter and the area of a circumscribed regular octagon. Use the result of Ex. 2, § 476.

479. Regular polygons in general. We have seen that as soon as the side of any inscribed regular polygon is known, the sides of inscribed and circumscribed regular polygons of double the number of sides may be found. Thus, beginning with the square, the sides of regular polygons of 4, 8, 16, 32, $\cdots$, 2^n sides may be found; beginning with the equilateral triangle, the sides of regular polygons of 3, 6, 12, 24, $\cdots$, 3×2^n sides may be found.

MEASUREMENT OF THE CIRCLE

480. Consider regular polygons $ABCDEF$ and $A'B'C'D'E'F'$, respectively, inscribed and circumscribed about a circle. Imagine the number of sides indefinitely increased, by any law — as by successive doubling. Then it is clear that the successive perimeters of the inscribed polygons become greater and greater (why?), and the successive perimeters of the circumscribed polygons become smaller and smaller (why?). Hence the difference between the perimeters of the inscribed and circumscribed polygons becomes less and less, though never zero (why?).

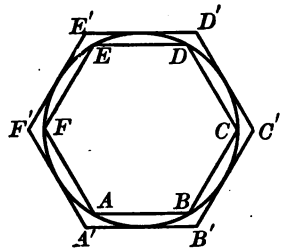


FIG. 217.

481. FUNDAMENTAL PROPOSITION. *We shall assume that the perimeters of regular polygons inscribed in and circumscribed about a circle approach a common limiting value, when the number of sides is indefinitely increased; and also that the areas of these polygons approach a common limiting value.*

482. The circumference and the area of a circle. The common limit of the perimeters of regular polygons inscribed in, and circumscribed about, a circle, when the number of sides is indefinitely increased is called *the length of the circle*, or its *circumference;* and the common limit of their areas is called the *area of the circle.*

483. Problem. *Given R, the radius of a circle, to find its circumference.*

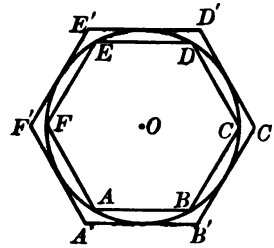


Fig. 218.

Solution: 1. Find successively the perimeters of inscribed and circumscribed regular polygons of 6, 12, 24, ··· sides, and tabulate the results as shown in the table. (The student is not expected to make these computations.)

Number of Sides.	Perimeter of Inscribed Polygon.	Perimeter of Circumscribed Polygon.
6	$3.0000000 \times 2R$	$3.4641016 \times 2R$
12	$3.1058285 \times 2R$	$3.2153903 \times 2R$
24	$3.1326286 \times 2R$	$3.1596599 \times 2R$
48	$3.1393502 \times 2R$	$3.1460862 \times 2R$
96	$3.1410320 \times 2R$	$3.1427146 \times 2R$
192	$3.1414526 \times 2R$	$3.1418730 \times 2R$
384	$3.1415577 \times 2R$	$3.1416627 \times 2R$
768	$3.1415847 \times 2R$	$3.1416101 \times 2R$
1536	$3.1415904 \times 2R$	$3.1415970 \times 2R$

2. We conclude that the circumference, C, of the circle, which by definition is the common limiting value of these two series of perimeters, is equal to $2R$ times a certain *fixed* number, which lies between

$$3.1415904 \text{ and } 3.1415970$$

3. This number, which expresses the ratio, $\frac{C}{2R}$, of the circumference of a circle to its diameter, is universally denoted by the Greek letter π (pronounced "pi").

4. $\therefore C = 2\pi R$, where $\pi = 3.1416$ approximately.

Remark. Instead of beginning with the regular hexagon, we might just as well begin with the square and compute, successively, the perimeters of inscribed and circumscribed polygons of 4, 8, 16, 32, 64, 128, ··· sides.

484. The number π. The number $\pi = \frac{C}{2R} = 3.14159\cdots$, can be proved by the methods of higher mathematics to be an incommensurable number. Its value has been computed to over seven hundred decimal places. We give here its value to the first ten decimal places:

$$\pi = 3.1415926535.$$

485. PROBLEM. *Given R, the radius of a circle, to find its area.*

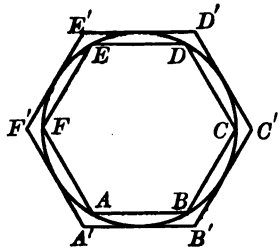


FIG. 219.

SOLUTION: 1. Let $A'B'C'\cdots$ be a regular polygon circumscribed about the circle. Denote the perimeter of this polygon by P' and its area by S'; further, denote the circumference of the circle by C and its area by A.

2. Now, $S' = \frac{1}{2} R \cdot P'$. § 361

3. As the number of sides of the polygon is increased, the perimeter P' approaches the circumference C of the circle while the area S' of the polygon approaches the area S of the circle. § 382

4. $\therefore$ the ratio $\frac{S'}{P'}$ can be made to differ as little as we please from the ratio $\frac{S}{C}$, by taking a sufficiently large number of sides.

5. But $\frac{S'}{P'} = \frac{1}{2} R =$ a given number, not depending on the number of sides of the polygon.

6. $\therefore \frac{S}{C} = \frac{1}{2} \cdot R.$

7. $\therefore S = \frac{1}{2} R \cdot C = \frac{1}{2} R \cdot 2\pi R = \pi R^2.$

486. COROLLARY 1. *The area of a circle is equal to one half the product of its circumference by its radius.*

487. COROLLARY 2. *The area of a circle is equal to the square of its diameter by one fourth π.*

488. The method of limits. The method by means of which we have computed the circumference and the area of a circle in terms of its radius is called "a limiting process," or "the method of limits." It was probably first used by Archimedes under the name of "the method of exhaustion." The method consists essentially in confining (or capturing) a number whose value is to be determined between two variable numbers whose difference can be made as small as we please, though not necessarily zero, and whose values can be computed by known methods.

The method is of great importance and is the basis of all methods for computing the lengths of, and the areas inclosed by, curved lines; as well as the areas of, and the volumes inclosed by, curved surfaces.

489. EXERCISES

1. In pioneer days a common method for measuring distances was to tie a ribbon to a spoke on a wagon wheel and count the number of revolutions made in traversing the ground to be measured. Assuming the diameter of a wheel to be 38 inches, find the number of revolutions made in traversing one mile.

2. If the diameter of a bicycle wheel is 28 inches, what distance will it cover in making 100 revolutions?

3. Find the required length of a belt to run over two pulleys, each 24 inches in diameter, if the distance between their centers is 15 feet.

4. A circular pond has a diameter of 100 yards. Find the cost at $2 per square yard of constructing a cement walk 3 feet wide around the pond.

5. The piston of a steam engine is 8 inches in diameter. If the steam gauge registers a pressure of 120 pounds per square inch, find the total pressure on the piston.

MEASUREMENT OF ARCS OF A CIRCLE

490. The radian, *or the circular unit of angle.* For theoretical purposes the unit angle most frequently employed in mathematics is the *radian;* it is the angle subtended at the center of a circle by an arc equal in length to the radius.

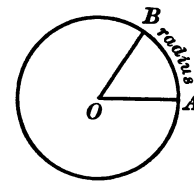


FIG. 220.

491. THEOREM. *Two central angles of a circle are proportional to their intercepted arcs.*

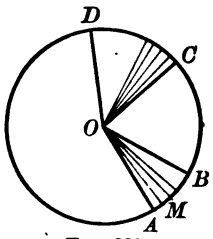


FIG. 221.

Given the circle with the central angles AOB and COD intercepted by the arcs AB and CD, respectively.

To prove that $\dfrac{\angle AOB}{\angle COD} = \dfrac{\text{arc } AB}{\text{arc } CD}$.

Proof. 1. If possible, find a common measure of arcs AB and CD, and suppose it is contained in arc AB m times and in arc CD n times.

2. Then $\dfrac{\text{arc } AB}{\text{arc } CD} = \dfrac{m}{n}$. § 338

3. Draw radii to the points of division on arcs AB and CD.

4. These radii divide $\angle s$ AOB and COD into m and n equal angles, respectively. Why?

5. $\therefore \dfrac{\angle AOB}{\angle COD} = \dfrac{m}{n}$. § 338

6. $\therefore \dfrac{\angle AOB}{\angle COD} = \dfrac{\text{arc } AB}{\text{arc } CD}$. Why?

REMARK. If arcs AB and CD are incommensurable, the above proof is not valid. Why? The reasoning for this case is analogous to that given in § 391. Let the student give it.

492. COROLLARY 1. *The sum of the angles about a point in a plane is equal to 2π radians.*

For $\dfrac{C}{R} = 2\pi$ is the number of radians subtended by a whole circle.

493. COROLLARY 2. *An arc of a circle is equal to the product of the radius and the subtended central angle expressed in radians.*

Let $a =$ the length of the arc.

$R =$ the radius.

$\theta =$ the angle expressed in radians.

Then $\dfrac{a}{\text{arc subtended by 1 radian}} = \dfrac{\theta}{1 \text{ radian}}$. § 491

Or $\dfrac{a}{R} = \dfrac{\theta}{1}$. $\therefore a = R\theta$.

494. COROLLARY 3. *The radian is a constant, that is, it is the same angle in all circles, and is equal to* $\dfrac{360°}{2\pi}$, *or* 1 *radian* $= 57.3°$, *approximately.*

495. A sector of a circle. The figure formed by the radii OA and OB of a circle and the arc AB subtended by the central angle AOB is called a *sector*, and is read sector AOB. The angle AOB and its subtended arc AB are called, respectively, *the central angle* and *the arc of the sector.*

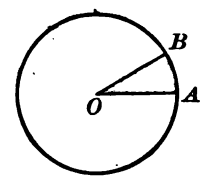


FIG. 222.

496. A segment of a circle. The figure formed by a chord of a circle and either of the intercepted arcs is called a *segment of a circle.*

Thus, the chord CD forms with the arcs CED and CFD the segments CED and CFD, respectively.

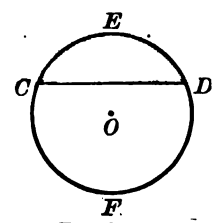


FIG. 223.

497. COROLLARY 4. *Two sectors of a circle are proportional to their central angles, or their arcs.*

498. COROLLARY 5. *The area of a sector is equal to one half the product of its arc by the radius.*

Let $K =$ the area of the sector.

$a =$ the length of its arc.

$R =$ the radius of the circle.

$S =$ the area of the circle.

Then $\frac{K}{S} = \frac{a}{2\pi R}$. § 497

$$\therefore K = \frac{a}{2\pi R} \times S = \frac{a}{2\pi R} \times \pi R^2 = \frac{aR}{2}.$$

499. EXERCISES

1. Find the angle subtended at the center of a circle by an arc $6\frac{1}{2}$ feet long, if the radius of the circle is $12\frac{1}{4}$ feet.

2. Find the area of a sector if the angle at the center is 35° and the radius of the circle is 40 inches.

3. Find the length of one degree of longitude at the equator, assuming the equatorial radius to be 3963 miles.

4. What is the distance between two points on the equator whose local times differ by 1 hour?

5. On a circle having a radius of 100 feet, an arc of 100 feet subtends what angle at the center, expressed in degrees? Expressed in radians?

6. If a flywheel makes 1000 revolutions a minute, how long does it take to turn through 1 radian?

7. Find the radius of a circle if an arc of 10 feet subtends an angle of 1 degree at the center.

8. Find the area of a sector whose central angle is 1 radian if the radius of the circle is 10 feet.

9. Find the area of a segment of a circle if its arc subtends an angle of 60° at the center and the radius of the circle is 100 feet.

10. The driving pulley on an engine is 13 feet in diameter, and makes 120 revolutions per minute. If it is belted to a 5-foot pulley on the main line shaft, find the speed of the latter.

11. An iron washer has an outside diameter of $1\frac{7}{8}$ inches, and the width of the washer is $\frac{1}{2}$ inch. Find the area of one side of the washer.

12. Two tangents to a circle form an angle of 120°. If the radius of the circle is 50, find the area of the figure formed by the tangents and the intercepted arc.

13. The radius of a circle is R. Find the radius of a circle twice as large; n times as large.

14. Find the radius of a circle equal to the sum of two circles whose radii are 6 and 8, respectively.

15. The radius of a circle is 100. Find the radii of two concentric circles which divide the given circle into three equal parts.

THREE-PLACE TABLE OF SINES, COSINES, AND TANGENTS

Deg.	Sin.	Cos.	Tan.	Deg.	Sin.	Cos.	Tan.
0	.000	1.000	.000	46	.719	.695	1.04
1	.017	1.000	.017	47	.731	.682	1.07
2	.035	.999	.035	48	.743	.669	1.11
3	.052	.999	.052	49	.755	.656	1.15
4	.070	.998	.070	50	.766	.643	1.19
5	.087	.996	.087	51	.777	.629	1.23
6	.105	.995	.105	52	.788	.616	1.28
7	.122	.993	.123	53	.799	.602	1.33
8	.139	.990	.141	54	.809	.588	1.38
9	.156	.988	.158	55	.819	.574	1.43
10	.174	.985	.176	56	.829	.559	1.48
11	.191	.982	.194	57	.839	.545	1.54
12	.208	.978	.213	58	.848	.530	1.60
13	.225	.974	.231	59	.857	.515	1.66
14	.242	.970	.249	60	.866	.500	1.73
15	.259	.966	.268	61	.875	.485	1.80
16	.276	.961	.287	62	.883	.469	1.88
17	.292	.956	.306	63	.891	.454	1.96
18	.309	.951	.325	64	.899	.438	2.05
19	.326	.946	.344	65	.906	.423	2.14
20	.342	.940	.364	66	.914	.407	2.25
21	.358	.934	.384	67	.921	.391	2.36
22	.375	.927	.404	68	.927	.375	2.48
23	.391	.921	.424	69	.934	.358	2.61
24	.407	.914	.445	70	.940	.342	2.75
25	.423	.906	.466	71	.946	.326	2.90
26	.438	.899	.488	72	.951	.309	3.08
27	.454	.891	.510	73	.956	.292	3.27
28	.469	.883	.532	74	.961	.276	3.49
29	.485	.875	.554	75	.966	.259	3.73
30	.500	.866	.577	76	.970	.242	4.01
31	.515	.857	.601	77	.974	.225	4.33
32	.530	.848	.625	78	.978	.208	4.70
33	.545	.839	.649	79	.982	.191	5.14
34	.559	.829	.675	80	.985	.174	5.67
35	.574	.819	.700	81	.988	.156	6.31
36	.588	.809	.727	82	.990	.139	7.12
37	.602	.799	.754	83	.993	.122	8.14
38	.616	.788	.781	84	.995	.105	9.51
39	.629	.777	.810	85	.996	.087	11.4
40	.643	.766	.839	86	.998	.070	14.3
41	.656	.755	.869	87	.999	.052	19.1
42	.669	.743	.900	88	.999	.035	28.6
43	.682	.731	.933	89	1.000	.017	57.3
44	.695	.719	.966	90	1.000	.000	
45	.707	.707	1.000				

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